Travelling Wave Solutions of the Heat Equation in an Unbounded Cylinder with a Non-Linear Boundary Condition

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Abstract

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. The existence of travelling wave solutions for the heat equation $\partial_t u - \Delta u = 0$ in the unbounded cylinder $\mathbb{R} \times \Omega$ subject to the nonlinear boundary condition $\frac{\partial u}{\partial n} = f(u)$ is investigated. Finding such a solution amounts to solving the semi-linear elliptic PDE

$$
\begin{cases}
\Delta u - c \partial_x u = 0 & \text{for } (x, y) \in \mathbb{R} \times \Omega \\
\frac{\partial u}{\partial n} = f(u) & \text{for } (x, y) \in \mathbb{R} \times \partial \Omega .
\end{cases}
$$

The main result is the existence of a non-trivial solution of (\ast) for a large class of nonlinearities $f$. Additionally, asymptotic behavior at $x = \pm \infty$ and regularity properties are established.

A variational approach is used. More specifically, a weak solution of (\ast) is found as the minimizer of a constrained minimization problem posed in the weighted Sobolev space $H^1_2(\mathbb{R} \times \Omega, e^{-x})$. Due to underlying domain $\mathbb{R} \times \Omega$ being unbounded, this problem suffers from a lack of compactness. The problem is solved using a suitable approximation.

Kurzfassung

Sei $\Omega \subset \mathbb{R}^2$ ein beschränktes Gebiet. Die Existenz von fortschreitenden Wellenlösungen der Wärmeleitungsgleichung in dem unbeschränkten Zylinder $\mathbb{R} \times \Omega$ mit der nicht-linearen Randbedingung $\frac{\partial u}{\partial n} = f(u)$ wird untersucht. Dieses Problem reduziert sich auf die semi-lineare elliptische partielle Differentialgleichung

$$
\begin{cases}
\Delta u - c \partial_x u = 0 & \text{für } (x, y) \in \mathbb{R} \times \Omega \\
\frac{\partial u}{\partial n} = f(u) & \text{für } (x, y) \in \mathbb{R} \times \partial \Omega .
\end{cases}
$$

Das Hauptresultat zeigt die Existenz von Lösungen der Gleichung (\ast) für eine große Klasse von Nichtlinearitäten $f$. Darüber hinaus wird das asymptotische Verhalten in $x = \pm \infty$ untersucht und Regularitätseigenschaften nachgewiesen.

Um die Existenz zu zeigen, werden direkte variationelle Methoden benutzt. Eine schwache Lösung der Gleichung (\ast) wird durch die Überführung der Gleichung in ein Minimierungsproblem mit Nebenbedingung gefunden. Das Minimierungsproblem wird in dem gewichteten Sobolev-Raum $H^1_2(\mathbb{R} \times \Omega, e^{-x})$ gestellt. Wegen der Unbeschränktheit des Zylinders tritt dabei das Problem der mangelnden Kompaktheit auf. Dies wird durch die Einführung eines geeigneten Approximationsproblems gelöst.
I express my deepest gratitude to my advisor Prof. Dr. Josef Bemelmans for his support. Further I also thank my colleagues at Institut für Mathematik, RWTH Aachen for many fruitful discussions. Finally, let me thank Prof. Dr. Wolfgang Marquardt for suggesting an exciting mathematical problem with an interesting physical background.
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1 Introduction

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain. Consider the heat equation in the unbounded cylinder \( \mathbb{R} \times \Omega \) with a non-linear dissipation condition on the boundary,

\[
\begin{cases}
\partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R} \times \Omega \\
\frac{\partial u}{\partial n} = f(u) & \text{on } \mathbb{R}^+ \times \mathbb{R} \times \partial \Omega.
\end{cases}
\]

(1.1)

In the following work the existence of non-trivial travelling wave solutions for the above problem is investigated. A travelling wave solution is a function \( u \) defined on \( \mathbb{R} \times \Omega \) such that

\[
(t, x, y) \to u(x + ct, y) , \quad (t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega
\]

(1.2)

solves (1.1). More specifically, (1.2) represents a travelling wave in the \( x \)-direction with propagation speed given by the constant \( c \). Finding such a solution amounts to solving the elliptic equation

\[
\begin{cases}
\Delta u - c \partial_x u = 0 & \text{in } \mathbb{R} \times \Omega \\
\frac{\partial u}{\partial n} = f(u) & \text{on } \mathbb{R} \times \partial \Omega.
\end{cases}
\]

(1.3)

The propagation speed \( c \) is typically not prescribed. Hence the problem is correctly formulated as finding a solution pair \((c, u)\) of (1.3).

A class of non-linearities \( f \) characterized by \( f(0) = 0 \) and \( f(s) \geq 0 \), \( s \in \mathbb{R} \) are considered. Due to the physical background of the problem, non-linearities vanishing only at 0 are of special interest and will be in focus throughout the work.

While semi-linear reaction diffusion equations in cylinders have been studied over the years, few results have been obtained for problems with non-linear boundary conditions of type \( \frac{\partial u}{\partial n} = f(u) \). Most of the existing methods rely on the existence of at least two trivial solutions. Such methods typically recover a non-trivial solution as a connection, in some sense, between the trivial ones. In (1.3) the trivial solutions are simply the constants corresponding to the vanishing points of \( f \). Thus in the case of a non-linearity \( f \) vanishing only at 0 only a single trivial solution is involved. This complicates the use of the existing methods. Furthermore, the underlying domain \( \mathbb{R} \times \Omega \) of the problem is unbounded causing a lack of compactness which complicates the use of variational and topological methods. Note that in order to properly define a travelling wave solution, it is essential that the domain is unbounded in at least one dimension.

The main result in the following is the existence of a non-trivial solution of (1.3) for a large class of non-linearities \( f \) vanishing only at 0. A variational approach is used. Furthermore, regularity properties and asymptotic behavior at \( \pm \infty \) of the solution are investigated.
1.1 Background

The heat equation with a non-linear dissipation condition on the boundary appears in the study of transient boiling processes. To illustrate this, we consider a solid material being heated up (see Figure 1). Assume a part of the surface is in contact with a liquid of lower temperature than the solid. In this case a transport of heat takes place from the solid into the liquid. The heat flux on the boiling surface between the liquid and the solid material is described by a so-called boiling curve. More specifically, letting $f$ denote the boiling curve, the temperature $u$ of the liquid satisfies the condition $\frac{\partial u}{\partial n} = f(u)$ on the boiling surface. In the liquid itself the temperature satisfies the heat equation. We are thus led to equation (1.1).

Figure 2 shows a typical boiling curve characterized by a decline immediately after...
the critical point. The critical point is simply the boiling point of the liquid. Once the
temperature reaches this point, gas bubbles will arise on the boiling surface. The decline
in the boiling curve reflects the lower heat conductivity between a solid material and gas
as compared to a liquid. As the temperature further increases, the boiling process enters
a new phase where the gas bubbles part very quickly from the surface. As a consequence,
the boiling curve again starts to grow steadily.

Experiments indicate (see [Blu98]) the existence of so-called heat waves in the setting
described above. Heat waves are simply travelling temperature waves. This leads us to
equation (1.3).

Considering (1.3) as a model of the problem, we have a system with a positive in-flow of
energy. Under such circumstances, a travelling wave solution should have a positive limit
at one end of the cylinder. As will be shown, this indeed turns out to be the case for the
solutions we find.

1.2 Related results on travelling waves

Travelling wave solutions of semi-linear reaction diffusion equations in unbounded
cylinders have been studied by a number of authors. In the one-dimensional case early
results date back to the famous paper [KPP37] by Kolmogorov, Petrovskii, and Piskunov.
In later work, in particular by Aronson and Weinberger (see [AW75] and [AW78]) and
Fife and McLeod (see [FM77]), these results are extended.

In higher dimensions, fewer results exist. The methods developed in the one-
dimensional case all rely on ordinary differential equation arguments and do not extend
easily to higher dimensions. One problem with higher dimensional cylinders is that
boundary values have to be taken into consideration. At the present time, results seem to
exist only for homogeneous Dirichlet and Neumann boundary values.

One of the first results for Neumann boundary values was obtained in the two-
dimensional strip

\[ S = \{ (x, y) \in \mathbb{R}^2 \mid x \in \mathbb{R}, 0 < y < L \} \]

by Henri Berestycki and Bernard Larrouturou. In the paper [BL89] they show existence of
solutions of the problem

\[
\begin{align*}
\Delta u - c \alpha(y) \partial_x u + f(u) &= 0 \quad \text{in } S \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial S \\
u(-\infty, y) &= 0, \quad u(\infty, y) = 1 \quad \text{for } 0 \leq y \leq L
\end{align*}
\]

(1.4)

for a class of non-linearities \( f \) arising in flame propagation models in combustion theory.
More specifically, they prove existence of solutions when \( f \) satisfies \( f(s) = 0 \) for \( s \in [0, \Theta] \),
1.2 Related results on travelling waves

$f(s) > 0$ for $s \in (\Theta, 1)$, and $f(1) = 0$. Note that (1.4) is the equation governing the travelling wave solutions (see (1.2)) of the associated reaction diffusion equation

$$\alpha(y) \partial_t u - \Delta u = f(u).$$

Berestycki and Larrouturou first solve the problem in finite rectangles using a topological degree argument. Considering larger and larger rectangles, they gain a sequence of such solutions. The assumption $f(0) = f(1) = 0$ implies by the maximum principle and Hopf’s Lemma the apriori estimate $0 \leq u \leq 1$ for all elements. By another application of maximum principles and comparison arguments, they show exponential decay estimates of the solutions. Finally, exploiting the apriori boundedness and the decay estimates, a solution of (1.4) is found as the limit of the sequence by a compactness argument.

In the later work [BLL90], by the same authors and P.L. Lions, the results from [BL89] are generalized and extended into arbitrary dimensions. In [BN92] the same methods are developed even further by Berestycki and Louis Nirenberg. More specifically, Berestycki and Nirenberg prove existence of solutions in $n$-dimensional cylinders $\mathbb{R} \times \Omega$ for the slightly more general problem

$$\begin{cases}
\Delta u - \beta(y, c) \partial_x u + f(u) = 0 & \text{in } \mathbb{R} \times \Omega \\
\frac{\partial u}{\partial n} = 0 & \text{on } \mathbb{R} \times \partial \Omega \\
u(-\infty, y) = 0, \quad u(\infty, y) = 1 & \text{for } y \in \Omega
\end{cases}$$

(1.5)

for the three classes of nonlinearities characterized in Figure 3. Importantly, the nonlinearities under consideration must still satisfy $f(0) = f(1) = 0$. The nonlinearity in case (B) is the same as the one considered in [BL89] originating from combustion theory. The nonlinearities in case (A) and (C) arise in problems of biology (population dynamics, gene developments, and epidemiology). As in [BL89], Berestycki and Nirenberg first prove existence in a finite cylinder. However, instead of a topological degree argument they prove existence using sub- and super-solutions and monotone iteration. The assumption
1.2 Related results on travelling waves

\[ f(0) = f(1) = 0 \] immediately delivers a sub- and a super-solution, namely the constants 0 and 1. Using a sliding domain argument they establish monotonicity properties and apriori bounds. The sliding domain method was developed by the same authors in [BN88] and further developed in [BN91]. It is essentially a modified moving plane argument. In addition to the apriori bounds, they also prove exponential decay estimates by comparison arguments and the maximum principle. For the the comparison argument, solutions of the linear "eigenvalue" problem

\[
\begin{align*}
-\Delta \varphi + a(y)\varphi &= (\lambda^2 - \lambda \beta(y))\varphi & \text{in } \Omega \\
\frac{\partial \varphi}{\partial n} &= 0 & \text{on } \partial \Omega
\end{align*}
\]

are used. Knowledge about the "eigenfunctions" of this problem plays an essential role in the proof and a lot of effort is put into analyzing them. Considering larger and larger cylinders, they finally obtain a sequence of solutions which are shown to converge to a solution of (1.5) by a compactness argument.

Note that the constants 1 and 0 solve the partial differential equations in both (1.4) and (1.5) when the non-linearity satisfies \( f(0) = f(1) = 0 \). Thus viewing (1.4) and (1.5) as evolution equations in the \( x \)-variable, one could pose the problems as finding a connection between the two "rest states" 0 and 1. Even though classical theory for evolution equations is not applied, it seems this viewpoint is the basis for some of the arguments used in the above papers. In particular in [BN92] where also the more general condition \( u(\infty, y) = v(y) \) with \( v \) being a solution of

\[
\begin{align*}
\Delta v + f(v) &= 0 & \text{in } \Omega \\
\frac{\partial v}{\partial n} &= 0 & \text{on } \partial \Omega
\end{align*}
\]

is treated. Such a \( v \) can be seen as a non-trivial "rest state" of (1.5).

Moving on to Dirichlet boundary values, one of the first results was again obtained in the two-dimensional strip and is due to Robert Gardner. In [Gar86] he shows existence of a non-trivial solution for the somewhat specialized problem

\[
\begin{align*}
\Delta u - c \partial_x u + u(1-u)(u-\alpha) &= 0 & \text{in } S \\
u &= 0 & \text{on } \partial S
\end{align*}
\]

with \( 0 < \alpha < \frac{1}{2} \). Gardner first considers a discretization of (1.6) which he solves using the Conley index. Letting the mesh tend to zero, he then obtains a solution of (1.6) by a compactness argument. In his application of the Conley index, Gardner fully exploits the evolutionary nature, in the \( x \)-variable, of (1.6).

Using some of the same ideas and methods developed by Berestycki, Larrouturou, Lions, and Nirenberg for the Neumann problem, Jose M. Vega has shown existence of solutions
for more general Dirichlet problems in higher-dimensional cylinders. In [Veg93] he solves the problem

\[
\begin{cases}
\Delta u + c \partial_x u + f(u) = 0 & \text{in } \mathbb{R} \times \Omega \\
u = 0 & \text{on } \mathbb{R} \times \partial \Omega \\
u(-\infty, y) = v_-, \quad u(\infty, y) = v_+ & \text{for } y \in \Omega
\end{cases}
\]

with \(v_-\) and \(v_+\) being solutions of

\[
\begin{cases}
\Delta v + f(v) = 0 & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega
\end{cases}
\]

satisfying, among other conditions, \(v_- < v_+\). Using the terminology of evolution equations, \(v_-\) and \(v_+\) are ”rest states” of (1.7). All conditions on the non-linearity \(f\) are formulated implicitly in terms of how the potential

\[
H : H^1_2(\Omega) \to \mathbb{R} \quad , \quad H(u) := \int_{\Omega} |Du|^2 - 2 F(u) \, dy ,
\]

with

\[
F(u) = \int_0^u f(s) \, ds ,
\]

behaves on solutions of (1.8) compared to \(H(v_-)\) and \(H(v_+)\). Similar to the method used in [BN92], existence is first shown in finite cylinders using sub- and super-solutions and monotone iteration. In this case \(v_-\) and \(v_+\) are used as sub- and super-solution, respectively. In the usual manner, a sequence of solutions is then obtained by considering larger and larger cylinders. Using the sliding domain method from [BN88], monotonicity properties for the elements are established. Finally, using the implicit assumptions on \(f\), \(v_-\), and \(v_+\) together with a compactness argument, a solution of (1.7) is found as the limit of the sequence.

Problem (1.7) has also been studied by Steffen Heinze. In [Hei88] (which dates back further than [Veg93]) he shows existence of a non-trivial solution \(u\) for non-linearities \(f\) satisfying \(f(0) = f(1) = 1\) and \(f'(1) < 1\). In contrast to [Veg93], he does so without prescribing the asymptotic ”rest states” \(v_-\) and \(v_+\). Aposteriori, however, he shows \(u(-\infty, y) = 0\) and \(u(\infty, y) = v\) for a solution \(v\) of (1.8). His proof is twofold, being substantially different depending on whether \(f'(0) < \mu\) or \(f'(0) > \mu\) where \(\mu\) is the first eigenvalue of the negative Laplace operator in \(\Omega\) with Dirichlet boundary values. In the case \(f'(0) < \mu\), a variational method is used to show existence in a finite cylinder. More specifically, he finds solutions in \(D_L := (0, L) \times \Omega\) of

\[
\begin{cases}
\Delta u - c \partial_x u + f(u) = 0 & \text{in } D_L \\
u = 0 & \text{on } \partial D_L
\end{cases}
\]
as minimizers of the functional

\[ I(u) := \int_{D_L} e^{-cx} \left( \frac{1}{2} |Du|^2 - F(u) \right) \, d(x,y) \]  

in \( H^2_s(D_L) \). Before doing so, he explicitly chooses an appropriate constant \( c \). Letting \( L \) tend to infinity, he then gains a sequence of solutions on finite cylinders. Using a moving plane type argument similar to the sliding domain method, he proves exponential decay estimates. By yet another moving plane argument, monotonicity properties are established. Furthermore, apriori bounds are established using the maximum principle and the assumption \( f(0) = f(1) = 0 \). Using these bounds and decay estimates, he then shows that the sequence converges to a solution of (1.7) by a compactness argument. The monotonicity properties ensure that this solution is non-trivial. In the case \( f'(0) > \mu \), a monotone iteration method is used. A sub-solution is obtained by modifying a solution found in [AW78] of an equivalent one-dimensional equation. A super-solution is constructed using an eigenfunction of the Laplace operator in \( \Omega \) with Dirichlet boundary values. Applying the standard monotone iteration technique, solutions on finite cylinders \( D_L \) can then be found. Monotonicity properties and apriori bounds are established as in the previous case. Consequently, the sequence obtained by letting \( L \) tend to infinity is shown to converge to a non-trivial solution of (1.7).

## 1.3 Applying existing methods

When considering the nonlinear boundary condition \( \frac{\partial u}{\partial n} = f(u) \) as in (1.1), one is of course dealing with a problem substantially different in its nature than the semi-linear reaction diffusion equations with Dirichlet or Neumann boundary values described above. Nevertheless, there are similarities. In both situations one is faced with the same type of elliptic equation and the problem of overcoming the lack of compactness which is due to the unboundedness of the domain. Hence it makes sense to try and adapt the ideas and methods described in the previous section in the search for travelling wave solutions. However, these methods have some shortcomings when applied to (1.3), which we briefly discuss here.

Common to all the existing methods is the approximation by equivalent problems in bounded domains. When defining such an approximate problem in a finite cylinder, one has to define additional boundary conditions on the two end-pieces. This is no problem if working from the viewpoint of finding a connection, in the \( x \)-direction, between two "rest states". Then one simply considers \( u \) equal to the respective "rest state" as boundary condition on each end-piece. This is done in most of the papers above. When assuming \( f(0) = f(1) = 0 \), two "rest states" are immediately given by the constants 0 and 1. In the case of \( f \) vanishing only at 0, however, there are no two such obvious candidates. If in this case one would nevertheless choose to approximate by bounded domains, it would probably have to be done along the lines of the work by Berestycki and Nirenberg ([BN92])
1.4 New methods

and J.M. Vega ([Veg93]) using solutions of

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} &= f(u) \quad \text{on } \partial \Omega
\end{aligned}
\]  

(1.10)

as "rest states" on the two end-pieces. A solution \( u \) of (1.10), however, must satisfy

\[ f_\partial \int f(u) \, dS = 0. \]

Since we would like to consider non-linearities satisfying \( f(s) s > 0 \) for \( s > 0 \) we can rule out positive solutions of (1.10). Using a non-positive "rest state" of course prohibits the resulting travelling wave solution from being positive. Since we want to be able to obtain positive solutions, this approach is at best too restrictive. Due to the complicated nature of (1.10), it is at worst impossible.

Another obstacle in connection with the existing methods is the widespread use of the maximum principle and Hopf’s Lemma. When working with the boundary condition \( \frac{\partial u}{\partial n} = f(u) \) one has, in contrast to Neumann or Dirichlet boundary conditions, little information available on the boundary which can be exploited in an application of the maximum principle or Hopf’s Lemma. Especially when doing comparison arguments, say on \( u \) and \( v \), this problem becomes apparent. Since resolving the sign of \( f(u) - f(v) \) is very delicate when \( f \) resembles a boiling curve for example, applying Hopf’s Lemma becomes problematic. In all the existing methods, Hopf’s Lemma and the maximum principle play an essential role not only in establishing decay estimates and apriori bounds, but also in the sliding domain method and moving plane arguments.

1.4 New methods

We consider a bounded domain \( \Omega \subset \mathbb{R}^2 \) with a \( C^3 \)-boundary. The restriction that \( \Omega \) be two-dimensional is for simplicity only. Most results can be extended without modification into arbitrary dimensional cylinders. Inspired by the work of Steffen Heinze ([Hei88]), we formulate (1.3) as a variational problem. For this purpose we introduce the weighted Sobolev space \( H^2_w(\mathbb{R} \times \Omega, e^{-x}) \). More precisely, letting \( L^2(\mathbb{R} \times \Omega, e^{-x}) \) denote the space

\[ L^2(\mathbb{R} \times \Omega, e^{-x}) := \{ u \in L^2_{\text{loc}}(\mathbb{R} \times \Omega) \mid \int_{\mathbb{R} \times \Omega} u^2 e^{-x} \, d(x,y) < \infty \} \]

we define \( H^2_w(\mathbb{R} \times \Omega, e^{-x}) \) as the space consisting of all functions \( u \in L^2(\mathbb{R} \times \Omega, e^{-x}) \) having weak derivatives \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2} \) also belonging to \( L^2(\mathbb{R} \times \Omega, e^{-x}) \),

\[ H^2_w(\mathbb{R} \times \Omega, e^{-x}) := \{ u \in L^2(\mathbb{R} \times \Omega, e^{-x}) \mid \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2} \in L^2(\mathbb{R} \times \Omega, e^{-x}) \}. \]

\( H^2_w(\mathbb{R} \times \Omega, e^{-x}) \) is equipped with the norm

\[
\|u\|_{H^2_w(\mathbb{R} \times \Omega, e^{-x})} := \left( \int_{\mathbb{R} \times \Omega} (|Du|^2 + u^2) e^{-x} \, d(x,y) \right)^{\frac{1}{2}}.
\]
We define on $H^1_2(\mathbb{R} \times \Omega, e^{-x})$ the functionals

\begin{equation}
E(u) := \frac{1}{2} \int_{\mathbb{R} \times \Omega} |Du|^2 e^{-x} \, d(x,y)
\end{equation}

and

\begin{equation}
J(u) := \int_{\mathbb{R} \times \Gamma} F(u) e^{-x} \, dS(y) \, dx ,
\end{equation}

whereby

\begin{equation}
F(u) := \int_{0}^{u} f(s) \, ds
\end{equation}

and $\Gamma := \partial \Omega$. We require $f(0) = 0$ and sufficient growth conditions on $f$ such that $J$ be well defined on $H^1_2(\mathbb{R} \times \Omega, e^{-x})$. Furthermore we define

\begin{equation}
C := \{ u \in H^1_2(\mathbb{R} \times \Omega, e^{-x}) \mid J(u) = 1 \}.
\end{equation}

Consider now the variational problem of minimizing $E$ over the class $C$,

\begin{equation}
E \longmapsto \text{Min in } C.
\end{equation}

A minimizer $u$, of this problem, satisfies the associated Euler-Lagrange equation

\begin{equation}
\int_{\mathbb{R} \times \Omega} Du \cdot Dv e^{-x} \, d(x,y) = \lambda \int_{\mathbb{R} \times \Gamma} f(u) v e^{-x} \, dS(y) \, dx
\end{equation}

for all $v \in H^1_2(\mathbb{R} \times \Omega, e^{-x})$. Here $\lambda$ is the corresponding Lagrange multiplier. If $u$ is sufficiently regular, partial integration in the above equation yields

\begin{equation}
\int_{\mathbb{R} \times \Gamma} \frac{\partial u}{\partial n} v e^{-x} \, dS(y) \, dx - \int_{\mathbb{R} \times \Omega} (\Delta u - \partial_x u) v e^{-x} \, d(x,y) = \lambda \int_{\mathbb{R} \times \Gamma} f(u) v e^{-x} \, dS(y) \, dx
\end{equation}

for all $v \in H^1_2(\mathbb{R} \times \Omega, e^{-x})$. By the Fundamental Lemma in the Calculus of Variations, $u$ then satisfies

\begin{equation}
\begin{cases}
\Delta u - \partial_x u = 0 & \text{in } \mathbb{R} \times \Omega \\
\frac{\partial u}{\partial n} = \lambda f(u) & \text{on } \mathbb{R} \times \Gamma.
\end{cases}
\end{equation}

The Lagrange multiplier $\lambda$ can be shown to be strictly positive. Thus on the scaled domain $\Omega^* = \lambda \Omega$ the dilation of $u$ by $\frac{1}{\lambda}$, $\tilde{u}(x,y) := u(\frac{1}{\lambda}x, \frac{1}{\lambda}y)$, solves (1.3) with $c = \frac{1}{\lambda}$. Furthermore, the solution is non-trivial due to the side-constraint $J(u) = 1$. Consequently, we obtain a non-trivial travelling wave solution of (1.1) by solving (1.15).
Having now formulated the variational problem to solve, let us briefly discuss the choice of the functional space $H^1_2(\mathbb{R} \times \Omega, e^{-x})$. The exponential weight was introduced purely for technical reasons. More specifically, it was introduced in order to express equation (1.3) as the Euler-Lagrange equation of (1.15). Hence it is legitimate to ask whether or not it is reasonable to expect a travelling wave to lie in this space. Basically there are two types of travelling waves, solitary waves and travelling fronts. A solitary wave $u$ (see Figure 4) is characterized by $u(x, \cdot) \to 0$ for $x \to \pm \infty$. A travelling front (see Figure 5) is characterized by $u(x, \cdot) \to 0$ for $x \to -\infty$ and $u(x, \cdot) \to v$ for $x \to \infty$ for some positive limit $v$, possibly infinity. By the nature of the equation, we expect to find travelling front solutions. Travelling fronts are also the interesting ones from a physical point of view in connection with the transient boiling process described in Section 1.1. Assuming a positive propagation speed $c$, which will in fact turn out to be the case, we are thus led to expect a solution resembling the one illustrated in Figure 5. Due to the exponential weight, $H^1_2(\mathbb{R} \times \Omega, e^{-x})$ indeed contains such functions and is therefore an appropriate space in which to pose the problem.

Under suitable growth-conditions on $f$, we prove existence of a minimizer of problem (1.15). Furthermore, we show that a minimizer is sufficiently regular in order to integrate partially in (1.16). Consequently the existence of a non-trivial travelling wave solution in $H^1_2(\mathbb{R} \times \Omega, e^{-x})$ of (1.1) is established. Additionally, we show that the solution is a classical solution provided $f$ is sufficiently regular. Finally, it is shown that the solution
has the asymptotic characteristics of a travelling front.

We use direct methods to solve (1.15). Using a Poincaré-type inequality it can be shown that a minimizing sequence of (1.15) is bounded in $H^1_2(\mathbb{R} \times \Omega, e^{-x})$. Consequently a sub-sequence hereof converges weakly. By convexity in the gradient, $\mathcal{E}$ is weakly lower semi-continuous. Thus the energy of the weak limit is minimal and one only needs to show that the weak limit is admissible, i.e. satisfies the side-constraint $J(u) = 1$. When working with bounded domains, this would typically be done by using an embedding of the Sobolev-space into an appropriate $L^p$-space. Since the underlying domain $\mathbb{R} \times \Omega$ in (1.15) is unbounded, however, none of the embeddings of $H^1_2(\mathbb{R} \times \Omega, e^{-x})$ into useful $L^p$-spaces, for example $L^2(\mathbb{R} \times \Gamma, e^{-x})$, are compact. In other words, the problem suffers from a lack of compactness. The lack of compactness turns out to be the biggest obstacle to overcome when solving (1.15). In fact, as a consequence one can construct minimizing sequences with weak limits not satisfying the side-constraint. Thus in order to apply direct methods, we cannot rely on arbitrary minimizing sequences but have to construct a special one with appropriate compactness properties.

The existence of a minimizer for problem (1.15) is shown in Section 2. On several occasions we need regularity results for weak solutions of non-homogeneous Neumann problems. Since the results we need are not included in the standard regularity theory, we shall prove them in Section 3. The main results in Section 3 are formulated for general elliptic equations and is independent of Section 2.
1.5 Notation

- $\Omega$ always denotes a bounded domain. Unless otherwise specified, $\Omega$ is a subset of $\mathbb{R}^2$ with a $C^3$-smooth boundary $\Gamma$.

- Depending on the context, $(x, y)$ is an element of $\mathbb{R} \times \Omega$ or $\mathbb{R} \times \Gamma$ with $x \in \mathbb{R}$ and $y \in \Omega$ or $y \in \Gamma$.

- $L^2(\mathbb{R} \times \Omega, e^{-x})$ denotes the space

$$L^2(\mathbb{R} \times \Omega, e^{-x}) := \{ u \in L^2_{loc}(\mathbb{R} \times \Omega) \mid \int_{\mathbb{R} \times \Omega} u^2 \ e^{-x} \ d(x,y) < \infty \}.$$  

- Similarly we define

$$L^2(\mathbb{R} \times \Gamma, e^{-x}) := \{ u \in L^2_{loc}(\mathbb{R} \times \Gamma) \mid \int_{\mathbb{R} \times \Gamma} u^2 \ e^{-x} \ dS(y)dx < \infty \}.$$  

- $H^1_2(\mathbb{R} \times \Omega, e^{-x})$ is defined as the space consisting of all functions $u \in L^2(\mathbb{R} \times \Omega, e^{-x})$ having weak derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y_1}$, and $\frac{\partial u}{\partial y_2}$ also belonging to $L^2(\mathbb{R} \times \Omega, e^{-x})$,

$$H^1_2(\mathbb{R} \times \Omega, e^{-x}) := \{ u \in L^2(\mathbb{R} \times \Omega, e^{-x}) \mid \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2} \in L^2(\mathbb{R} \times \Omega, e^{-x}) \}.$$  

$H^1_2(\mathbb{R} \times \Omega, e^{-x})$ is equipped with the norm

$$\|u\|_{H^1_2(\mathbb{R} \times \Omega, e^{-x})} := \left( \int_{\mathbb{R} \times \Omega} (|Du|^2 + u^2) e^{-x} d(x,y) \right)^{\frac{1}{2}}.$$  

- The space $H^{1/2}_2(\mathbb{R} \times \Gamma, e^{-x})$ is defined as range of the trace operator

$$T : H^1_2(\mathbb{R} \times \Omega, e^{-x}) \to L^2(\mathbb{R} \times \Gamma, e^{-x}).$$  

Alternatively, one can define

$$H^{1/2}_2(\mathbb{R} \times \Gamma, e^{-x}) := \{ u \in L^2(\mathbb{R} \times \Gamma, e^{-x}) \mid u e^{-\frac{1}{2}x} \in H^{1/2}_2(\mathbb{R} \times \Gamma) \},$$  

with $H^{1/2}_2(\mathbb{R} \times \Gamma)$ being the usual fractional order Sobolev space.

- The space $H^{m}_{2,B-loc}(\Omega)$ is defined in Definition A.3 in Appendix A.

- The symbols $\rightarrow$ and $\rightharpoonup$ are used to denote strong and weak convergence, respectively.
2 Existence

In this section the existence of a minimizer of problem (1.15) is shown. As mentioned in the introduction, direct methods will be used. More specifically, we find the minimizer as the weak limit of a minimizing sequence. Due to the lack of compact embeddings, we construct explicitly a minimizing sequence with sufficient compactness properties. In addition to the existence, which is proved in Section 2.4, the asymptotic behavior of the solution is discussed in Section 2.5. Finally, in Section 2.6 we summarize the results and discuss possible extensions.

In order to construct the minimizing sequence, an approximation of problem (1.15) is considered. The approximation is based on a sequence of cut-off functions with respect to the unbounded $x$-direction of the cylinder $\mathbb{R} \times \Omega$. More specifically, we consider a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of real measurable functions satisfying $\varphi_n(x) \to 0$ for $x \to \pm \infty$ and $\varphi_n(x) \to 1$ for $n \to \infty$ (see Figure (6)). Using these functions we define on $H^1_2(\mathbb{R} \times \Omega, e^{-x})$ the functionals

$$J_n(u) := \int_{\mathbb{R} \times \Gamma} \varphi_n(x) F(u) e^{-x} dS(y) dx$$

and corresponding classes

$$\mathcal{C}_n := \{u \in H^1_2(\mathbb{R} \times \Omega, e^{-x}) | J_n(u) = 1\} .$$

of admissible elements. We then consider for each $n \in \mathbb{N}$ the variational problem of minimizing $\mathcal{E}$ over $\mathcal{C}_n$,

$$\mathcal{E} \mapsto \text{Min in } \mathcal{C}_n .$$

Since $J_n(u) \to J(u)$ as $n \to \infty$ for any $u \in H^1_2(\mathbb{R} \times \Omega, e^{-x})$, problem (2.2) can be seen as an approximation of (1.15).

We first establish existence of a minimizer $u_n$ in (2.2) for each $n \in \mathbb{N}$. Then we show that the elements of the hereby obtained sequence $\{u_n\}_{n=1}^{\infty}$ can be perturbed such that the resulting sequence is a minimizing sequence of (1.15). Additionally, we establish
exponential decay estimates using a representation formula. As it turns out, these decay estimates ensure enough compactness properties in order for the weak limit of the sequence to be admissible and thereby a proper minimizer of (1.15).

An advantage of approximating (1.15) by problem (2.2) is that we stay in the domain \( \mathbb{R} \times \Omega \). In other words, we avoid having to work with bounded cylinders and thereby the problem of defining boundary values on the end-pieces. Furthermore, the decay estimates are established without the use of maximum principles. Thus we overcome the obstacles preventing us from applying the existing methods described in the introduction.

### 2.1 Lack of compactness

Here we briefly discuss the problem of using direct methods without compact embeddings and why in the case of problem (1.15) the approximation by (2.2) is a reasonable way of overcoming it.

The lack of compactness we are facing in (1.15) is due to the unboundedness of the domain \( \mathbb{R} \times \Omega \). As a consequence hereof, we do not have any compact embeddings of the Sobolev space, in which the problem is posed, into a corresponding \( L^p \)-space. The lack of compact embeddings implies that a weakly convergent sequence in the Sobolev space does not necessarily converge strongly in a corresponding \( L^p \)-space. In order to illustrate this problem in connection with direct methods, we consider an example.

Define for \( 2 \leq q < 6 \) the functional

\[
I : \dot{H}^1_2(\mathbb{R} \times \Omega) \to L^q(\mathbb{R} \times \Omega), \\
I(u) := \int_{\mathbb{R} \times \Omega} |Du|^2 + u^2 \, d(x, y)
\]

and consider the minimization problem

\[
(2.3) \quad I \longmapsto \text{Min in } \mathcal{A} := \{ u \in \dot{H}^1_2(\mathbb{R} \times \Omega) \mid \|u\|_{L^q(\mathbb{R} \times \Omega)} = 1 \}.
\]

This is the problem of finding a function realizing the best Sobolev constant of the embedding \( \dot{H}^1_2(\mathbb{R} \times \Omega) \hookrightarrow L^q(\mathbb{R} \times \Omega) \). Existence of a minimizer in (2.3) can be shown (see for example Corollary 2.1 in [Cha99]). Consider now such a minimizer \( u \) and define for \( n \in \mathbb{N} \) the translation

\[
u_n(x, y) := u(x + n, y) \quad \forall (x, y) \in \mathbb{R} \times \Omega.
\]

Since \( I \) and the the norm \( \| \cdot \|_{L^q(\mathbb{R} \times \Omega)} \) are both translation invariant in the \( x \)-direction, also \( u_n \) is a minimizer in (2.3) and \( \{u_n\}_{n=1}^\infty \) obviously a minimizing sequence. Clearly

\[
\int_{\mathbb{R} \times \Omega} u_n \varphi \, d(x, y) \to 0 \text{ for } n \to \infty \quad \forall \varphi \in C_c^\infty(\mathbb{R} \times \Omega)
\]
2.1 Lack of compactness

Figure 7: Shifting of "mass" towards infinity.

and thus $u_n \rightharpoonup 0$ in $\dot{H}^1_2(\mathbb{R} \times \Omega)$. The weak limit 0, however, is not an admissible element, i.e. $0 \notin \mathcal{A}$. Hence $\{u_n\}_{n=1}^{\infty}$ is a minimizing sequence with a weak limit which is not a minimizer. This could, of course, not have occurred had $\dot{H}^1_2(\mathbb{R} \times \Omega)$ been compactly embedded in $L^q(\mathbb{R} \times \Omega)$.

As illustrated by this example, we cannot expect every weak limit of a minimizing sequence to be a proper minimizer when applying direct methods in a setting with an unbounded domain. This does not rule out the use of direct methods though. If only we avoid using minimizing sequences as above, where "mass" is shifted towards infinity, we can still hope to find a minimizer as the weak limit.

Consider again problem (1.15). As previously mentioned, we want to find a minimizer of this problem as the weak limit of a minimizing sequence consisting of minimizers of (2.2). In order to avoid losing "mass" at infinity, we have to ensure that such a sequence stays sufficiently concentrated on some bounded set. This will be accomplished by choosing functions $\varphi_n$ in the definition of $J_n$ which concentrate around 0. As a consequence, a minimizer of (2.2) will turn out to concentrate around 0 too. A sequence of such elements is therefore a good candidate for a minimizing sequence with no shifting of "mass" towards infinity. We will need to show, however, that this concentration property is sufficient in order to obtain admissibility of the weak limit. We refer to a concentration property like this as a compactness property.

Comprehensive work on variational problems in unbounded domains with lack of compactness has been done by P.L. Lions. In [Lio84a] and [Lio84b] he introduces the Concentration-Compactness Lemma which can be used to formally characterize the above described phenomenon of shifting of "mass" towards infinity in a minimizing sequence. Furthermore, he develops for variational problems the so-called Concentration-Compactness Principle, a condition ensuring the compactness of every minimal sequence up to translation. A large class of variational problems in unbounded domains can be solved by proving that they satisfy this condition. Problem (1.15), however, does not.

For the sake of completeness, we mention that the Concentration-Compactness Prin-
2.2 The approximating problem

The principle can be extended (see [Lio85a] and [Lio85b]) to problems where the lack of compactness is caused by a constraint with a critical growth as opposed to an unbounded domain. This would, for example, be the case in a constrained minimization problem in $H^1_2(\Omega)$ with the side-constraint $\|u\|_2 = 1$. Here $2^*$ denotes the critical exponent of the Sobolev Embedding of $H^1_2(\Omega)$ into $L^q(\Omega)$. The lack of compactness in such a problem refers to this Sobolev Embedding not being compact.

2.2 The approximating problem

We now investigate the existence of a minimizer for the approximating problem (2.2). Existence is proved in the case when $\vartheta$ vanishes at infinity and $f$ satisfies certain growth conditions. Furthermore, a bound on the Lagrange multiplier of the associated Euler-Lagrange equation is established.

Essential to the proof is the following Poincaré-type inequality, which holds for functions in the space $H^1_2(\mathbb{R} \times \Omega, e^{-x})$ and ensures coercivity of the energy functional $\mathcal{E}$ with respect to the $H^1_2(\mathbb{R} \times \Omega, e^{-x})$ norm.

Lemma 2.1. (Poincaré-type inequality)

\[
\int_{\mathbb{R} \times \Omega} u^2 e^{-x} \, d(x, y) \leq 4 \int_{\mathbb{R} \times \Omega} |\partial_x u|^2 e^{-x} \, d(x, y)
\]

for all $u \in H^1_2(\mathbb{R} \times \Omega, e^{-x})$.

Proof.

Let $u \in H^1_2(\mathbb{R} \times \Omega, e^{-x})$. Choose a sequence of continuously differentiable functions $\{u_n\}_{n=1}^{\infty}$ from $H^1_2(\mathbb{R} \times \Omega, e^{-x})$ with bounded support such that $u_n \to u$ in $H^1_2(\mathbb{R} \times \Omega, e^{-x})$. For any fixed $y \in \Omega$ one has

\[
\begin{align*}
0 \leq & \int_{-\infty}^{\infty} \left( \partial_x [u_n e^{-\frac{x}{2}}] \right)^2 \, dx \\
= & \int_{-\infty}^{\infty} |\partial_x u_n|^2 e^{-x} \, dx + \frac{1}{4} \int_{-\infty}^{\infty} u_n^2 e^{-x} \, dx - \int_{-\infty}^{\infty} u_n \partial_x u_n e^{-x} \, dx.
\end{align*}
\]

By partial integration and using the fact that $u_n$ has bounded support, the last integral in (2.5) evaluates to

\[
\int_{-\infty}^{\infty} u_n \partial_x u_n e^{-x} \, dx = \left[ \frac{1}{2} u_n^2 e^{-x} \right]_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} u_n^2 e^{-x} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} u_n^2 e^{-x} \, dx.
\]
Hence by (2.5)
\[ 0 \leq \int_{\mathbb{R}} |\partial_x u_n|^2 e^{-x} \, dx - \frac{1}{4} \int_{\mathbb{R}} u_n^2 e^{-x} \, dx. \]
Integrating over $\Omega$ yields
\[ \int_{\mathbb{R} \times \Omega} u_n^2 e^{-x} \, d(x,y) \leq 4 \int_{\mathbb{R} \times \Omega} |\partial_x u_n|^2 e^{-x} \, d(x,y). \]
Letting $n \to \infty$ proves the lemma. 

Now let $\vartheta : \mathbb{R} \to \mathbb{R}$ be a real measurable function satisfying
\[ 0 \leq \vartheta(x) \leq 1, \quad \vartheta(x) \to 0 \text{ for } |x| \to \infty. \]
On the nonlinearity $f$ we impose the conditions
\[ f \in C^1(\mathbb{R}), \quad f(0) = 0, \quad |f'| \leq k, \quad 0 \leq f(s) \quad \forall s \in \mathbb{R}, \]
with $k$ being a positive constant. We consider the energy functional
\[ \mathcal{E}(u) := \frac{1}{2} \int_{\mathbb{R} \times \Omega} |Du|^2 e^{-x} \, d(x,y) \]
and define
\[ J_{\vartheta}(u) := \int_{\mathbb{R} \times \Gamma} \vartheta(x) F(u) e^{-x} \, dS(y) \, dx. \]
Here and in the following we always have
\[ F(u) := \int_0^u f(s) \, ds. \]
Put
\[ \mathcal{C}_\vartheta := \{ u \in H_2^1(\mathbb{R} \times \Omega, e^{-x}) \mid J_{\vartheta}(u) = 1 \}. \]
We now consider the problem of minimizing $\mathcal{E}$ over $\mathcal{C}_\vartheta$,
\[ \mathcal{E} \longmapsto \text{Min in } \mathcal{C}_\vartheta. \]
Since this problem is posed in the same functional space, $H_2^1(\mathbb{R} \times \Omega, e^{-x})$, as the original problem (1.15), it suffers from the same lack of compactness. However, the presence of $\vartheta$ in the definition of $J_{\vartheta}$ penalizes (note that $F \geq 0$ due to (2.7)) growth away from 0 for admissible elements with respect to $\mathcal{E}$. In other words, the more an element of $\mathcal{C}_\vartheta$ concentrates away from 0, the more it has to grow in order to satisfy the side-constraint $J_{\vartheta}(u) = 1$. Consequently, the elements of a minimizing sequence will concentrate around 0. As it turns out, this property ensures enough compactness in order for the weak limit of any subsequence hereof to be admissible and hence a proper minimizer. Thus we have the following theorem.
Theorem 2.2. Let $f$ be a real function satisfying (2.7) and $\vartheta$ a real measurable function satisfying (2.6). Then there exists a minimizer $u$ for $E$ over the class $C_\vartheta$.

Proof.

Let $\{u_n\}_{n=1}^\infty$ be a minimizing sequence for $E$ over the class $C_\vartheta$. Using Lemma 2.1 one has

\[
\int_{\mathbb{R} \times \Omega} \left( u_n^2 + |Du_n|^2 \right) e^{-x} \, d(x,y) \leq 4 \int_{\mathbb{R} \times \Omega} |\partial_x u_n|^2 e^{-x} \, d(x,y) + \int_{\mathbb{R} \times \Omega} |Du_n|^2 e^{-x} \, d(x,y) \leq 10 E(u_n).
\]

Since $\{u_n\}_{n=1}^\infty$ is a minimizing sequence, $\{E(u_n)\}_{n=1}^\infty$ is bounded. Hence it follows from (2.9) that $\{u_n\}_{n=1}^\infty$ is bounded in $H^1_2(\mathbb{R} \times \Omega, e^{-x})$. Due to the reflexivity of the (Hilbert)-space $H^1_2(\mathbb{R} \times \Omega, e^{-x})$, there exists a subsequence of $\{u_n\}_{n=1}^\infty$ (which for the sake of simplicity will still be denoted by $\{u_n\}_{n=1}^\infty$) converging weakly,

\[ u_n \rightharpoonup u \quad \text{for } n \to \infty \quad \text{in } H^1_2(\mathbb{R} \times \Omega, e^{-x}), \]

to a function $u \in H^1_2(\mathbb{R} \times \Omega, e^{-x})$. We now show that $J_\vartheta(u_n) \to J_\vartheta(u)$ for $n \to \infty$.

Let $\varepsilon > 0$ be given. Since $f$ satisfies (2.7), one has by Taylor-expansion on $F$ that

\[
|F(u_n) - F(u)| = |f(u) (u - u_n) + \frac{1}{2} f'(\xi(u, u_n)) (u - u_n)^2|
\leq |f'(\eta(0, u)) u (u - u_n)| + \frac{1}{2} k (u - u_n)^2
\leq k |u| |u - u_n| + \frac{1}{2} k (u - u_n)^2.
\]

It follows that

\[
|J_\vartheta(u_n) - J_\vartheta(u)|
\leq \int_{\mathbb{R} \times \Gamma} \vartheta(x) |F(u_n) - F(u)| e^{-x} \, dS(y)dx
\leq k \int_{\mathbb{R} \times \Gamma} \vartheta(x) |u| |u - u_n| e^{-x} \, dS(y)dx + \\
+ \frac{1}{2} k \int_{\mathbb{R} \times \Gamma} \vartheta(x) (u - u_n)^2 e^{-x} \, dS(y)dx
\leq k \left( \int_{\mathbb{R} \times \Gamma} \vartheta(x) u^2 e^{-x} \, dS(y)dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R} \times \Gamma} \vartheta(x) (u - u_n)^2 e^{-x} \, dS(y)dx \right)^{\frac{1}{2}}
\leq \frac{1}{2} k \int_{\mathbb{R} \times \Gamma} \vartheta(x) (u - u_n)^2 e^{-x} \, dS(y)dx.
\]

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Now choose $b \in \mathbb{R}$, $b > 0$ such that $\vartheta(x) < \varepsilon$ for $|x| \geq b$. Consider the restriction operator

$$R^1_b : H^1_2(\mathbb{R} \times \Omega, e^{-x}) \to H^1_2((-b, b) \times \Omega),$$

the trace operator

$$T_b : H^1_2((-b, b) \times \Omega) \to L^2(\partial((-b, b) \times \Omega)),$$

and the restriction operator

$$R^2_b : L^2(\partial((-b, b) \times \Omega)) \to L^2((-b, b) \times \Gamma).$$

Being restriction operators, $R^1_b$ and $R^2_b$ are clearly continuous. Furthermore, since the domain $(-b, b) \times \Omega$ is bounded, $T_b$ is compact (see §7-8 in [Wlo87]). It follows that the composition $S_b := R^2_b \circ T_b \circ R^1_b$ of these operators is compact. Applying $S_b$ to $\{u_n\}_{n=1}^{\infty}$ hence yields $u_n \to u$ strongly in $L^2((-b, b) \times \Gamma)$. Thus for sufficiently large $n$ one has

$$\int_{\mathbb{R} \times \Gamma} \vartheta(x) (u - u_n)^2 \, e^{-x} \, dS(y) dx$$

$$= \int_{-b}^{b} \int_{\Gamma} \vartheta(x) (u - u_n)^2 \, dS(y) dx + \int_{\mathbb{R} \setminus [-b, b]} \int_{\Gamma} \vartheta(x) (u - u_n)^2 \, dS(y) dx$$

$$\leq \varepsilon + \varepsilon \int_{\mathbb{R} \times \Gamma} (u - u_n)^2 \, dS(y) dx .$$

By the boundedness of the trace operator $T : H^1_2(\mathbb{R} \times \Omega, e^{-x}) \to L^2(\mathbb{R} \times \Gamma, e^{-x})$, it follows from the above that

$$\int_{\mathbb{R} \times \Gamma} \vartheta(x) (u - u_n)^2 \, dS(y) dx$$

$$\leq \varepsilon + \varepsilon C \int_{\mathbb{R} \times \Omega} ((u - u_n)^2 + |Du - D_{u_n}|^2) \, e^{-x} \, d(x, y),$$

for $n$ sufficiently large. Using Lemma 2.1 and the boundedness of $\{E(u_n)\}_{n=1}^{\infty}$ in (2.11) now yields

$$\int_{\mathbb{R} \times \Gamma} \vartheta(x) (u - u_n)^2 \, dS(y) dx \leq \varepsilon + \varepsilon C (E(u) + E(u_n)) \leq C \varepsilon ,$$

for sufficiently large $n$. It follows from (2.10) and (2.12) that $J_{\vartheta}(u_n) \to J_{\vartheta}(u)$ for $n \to \infty$.

Since $\{u_n\}_{n=1}^{\infty}$ is a sequence in $C_0$, one has $J_{\vartheta}(u_n) = 1$ for all $n \in \mathbb{N}$. It follows that $J_{\vartheta}(u) = 1$. Hence the weak limit $u$ is admissible. By convexity in the gradient, the functional $E$ is weakly lower semi-continuous. Consequently

$$E(u) \leq \liminf_{n \to \infty} E(u_n) = \inf_{v \in C_0} E(v) .$$
Thus $u$ has minimal energy and is therefore a minimizer for $\mathcal{E}$ over $C_\varphi$. \hfill \Box

2.2 The approximating problem

A minimizer $u$ for $\mathcal{E}$ over $C_\varphi$ satisfies the associated Euler-Lagrange equation

\begin{equation}
\int_{\mathbb{R} \times \Omega} D_u \cdot Dv \, e^{-x} \, d(x,y) = \lambda \int_{\mathbb{R} \times \Gamma} \vartheta(x) f(u) \, e^{-x} \, dS(y)dx ,
\end{equation}

for all $v \in H^1_0(\mathbb{R} \times \Omega, e^{-x})$. As mentioned in the beginning of this section, such minimizers will later be used to create a minimizing sequence for the main problem (1.15). The fact that each element satisfies (2.13) is then used to establish a pointwise bound depending, among other factors, on $\lambda$. For the further applications of this pointwise bound, the ability to control $\lambda$ therefore turns out to be important. In this context the following lemma is useful.

**Lemma 2.3.** Let $u$ be a minimizer of $\mathcal{E}$ over $C_\varphi$. Assume that $f$ satisfies (2.7) and

\begin{equation}
\vartheta F(s) \leq f(s) s \quad \forall s \in \mathbb{R}
\end{equation}

for some positive constant $\Theta > 0$. Then $u$ satisfies (2.13) and

\begin{equation}
0 < \lambda \leq \frac{2}{\Theta} \mathcal{E}(u) = \frac{2}{\Theta} \inf_{v \in C_\varphi} \mathcal{E}(v) .
\end{equation}

**Proof.**

As mentioned above, equation (2.13) is the Euler-Lagrange equation

$$\delta \mathcal{E}(u,v) = \lambda \delta J_\varphi(u,v)$$

corresponding to the variational problem of minimizing $\mathcal{E}$ over $C_\varphi$. It is a standard result from the Calculus of Variations that $u$ satisfies (2.13).

Putting $v = u$ in (2.13) now yields

\begin{equation}
0 \leq \int_{\mathbb{R} \times \Omega} |D u|^2 \, e^{-x} \, d(x,y) = \lambda \int_{\mathbb{R} \times \Gamma} \vartheta(x) f(u) \, e^{-x} \, dS(y)dx .
\end{equation}

Since $J_\varphi(u) = 1$ it follows that $u \neq 0$. Hence strict positivity holds in (2.16) and thus $\lambda \neq 0$. By (2.7), $f$ satisfies $0 \leq f(s) s$. The fact that $\vartheta \geq 0$ therefore implies

$$0 \leq \int_{\mathbb{R} \times \Gamma} \vartheta(x) f(u) \, e^{-x} \, dS(y)dx .$$

It follows that $\lambda > 0$.

Since $\lambda$ and $\vartheta$ are non-negative, applying assumption (2.14) in (2.16) yields

$$\lambda \Theta \int_{\mathbb{R} \times \Gamma} \vartheta(x) F(u) \, e^{-x} \, dS(y)dx \leq \int_{\mathbb{R} \times \Omega} |D u|^2 \, e^{-x} \, d(x,y) .$$
Thus
\[ \lambda \Theta J_\phi(u) \leq 2\mathcal{E}(u) . \]
Since \( J_\phi(u) = 1 \), the inequality (2.15) follows.

Remark 2.4. Since \( \Theta \) can be chosen arbitrarily small, condition (2.14) merely implies that \( f \) cannot converge to 0 at some point.

2.3 Representation formula and decay estimates

Each element of the minimizing sequence, which is to be constructed in order to find a minimizer for (1.15), will be a solution of an approximating problem of type (2.8). In this section, a representation formula for such solutions is established. Using this formula, pointwise exponential decay estimates are then shown.

A solution \( u \) of (2.8) satisfies, at least in the weak sense, an Euler-Lagrange equation of type

\[
\begin{cases}
\Delta u - \partial_x u = 0 & \text{in } \mathbb{R} \times \Omega, \\
\frac{\partial u}{\partial n} = g & \text{on } \mathbb{R} \times \Gamma,
\end{cases}
\]

with \( g \in L^2(\mathbb{R} \times \Gamma, e^{-x}) \) being the trace of an \( H^1_2(\mathbb{R} \times \Omega, e^{-x}) \)-function and hence in \( H^{1/2}_{2,B-loc}(\mathbb{R} \times \Gamma) \). By regularity theory (see Theorem 3.8) we thus have

\[
(2.18) \quad u \in H^1_2(\mathbb{R} \times \Omega, e^{-x}) \quad \text{and} \quad \forall N \in \mathbb{N} : \ u \in H^2_2((-N,N) \times \Omega) .
\]

The results in this section are established for all functions satisfying (2.17) and (2.18). Due to standard regularity theory for elliptic equations, such functions all belong to \( C^{\infty}(\mathbb{R} \times \Omega) \). Further note that functions satisfying (2.18) have normal derivatives on \( \mathbb{R} \times \Gamma \) at least in the trace sense, which is the way the boundary condition in (2.8) is to be understood.

In the following, \( y = (y_1, y_2, y_3) \) and \( \xi = (\xi_1, \xi_2, \xi_3) \) shall, depending on the context, denote points in \( \mathbb{R} \times \Omega \) or \( \mathbb{R} \times \Gamma \). Consider the function

\[
(2.19) \quad \varphi(y) = \frac{1}{|y|} e^{-\frac{1}{2}|y| - \frac{1}{2} y_1} , \quad y \in \mathbb{R}^3 \setminus \{0\} .
\]

\( \varphi \) satisfies

\[
(2.20) \quad \Delta \varphi + \partial_1 \varphi = 0 , \quad \text{for } y \in \mathbb{R}^3 \setminus \{0\}
\]

and hence is a fundamental solution for the elliptic operator in (2.17). Interestingly, \( \varphi \) satisfies exactly the right growth conditions in order for the convolution between \( \varphi \) and functions from \( L^2(\mathbb{R} \times \Omega, e^{-x}) \) to be well-defined in the classical sense. This property of \( \varphi \) makes it possible to establish the following representation formula.
Theorem 2.5. Let $u$ be a weak solution of (2.17) satisfying (2.18). Then

\begin{equation}
(2.21) \quad u(y) = \frac{1}{3\omega_3} \int_{\mathbb{R} \times \Gamma} g(\xi) \varphi(\xi - y) - u(\xi) \frac{\partial \varphi}{\partial n}(\xi - y) \, dS(\xi)
\end{equation}

for all $y \in \mathbb{R} \times \Omega$, with $\omega_3$ being the measure of the three-dimensional unit-ball.

Proof.

Fix $y \in \mathbb{R} \times \Omega$. Let $\varepsilon > 0$ be sufficiently small so that $B_\varepsilon(y) \subset \mathbb{R} \times \Omega$. Consider the derivatives of $\varphi$. One has

\begin{align*}
\partial_1 \varphi(y) &= P_1(y) e^{-\frac{1}{2}|y| - \frac{1}{2}y_1}, \quad P_1 \text{ continuous and bounded away from 0}, \\
\Delta \varphi(y) &= P_\Delta(y) e^{-\frac{1}{2}|y| - \frac{1}{2}y_1}, \quad P_\Delta \text{ continuous and bounded away from 0}.
\end{align*}

It follows that

\begin{equation}
(2.22) \quad \xi \rightarrow u(\xi) \Delta \varphi(\xi - y) = u(\xi) P_\Delta(\xi - y) e^{-\frac{1}{2}|\xi - y| - \frac{1}{2}(\xi_1 - y_1)} \\
= u(\xi) e^{-\frac{1}{2}\xi_1} P_\Delta(\xi - y) e^{-\frac{1}{2}|\xi - y| + \frac{1}{2}y_1} \in L^1(\mathbb{R} \times \Omega \setminus B_\varepsilon(y)).
\end{equation}

Similarly

\begin{equation}
(2.23) \quad \xi \rightarrow u(\xi) \partial_1 \varphi(\xi - y) \in L^1(\mathbb{R} \times \Omega \setminus B_\varepsilon(y))
\end{equation}

and

\begin{equation}
(2.24) \quad \xi \rightarrow \partial_1 u(\xi) \varphi(\xi - y) \in L^1(\mathbb{R} \times \Omega \setminus B_\varepsilon(y)).
\end{equation}

Since $u$ is a solution of (2.17) and belongs to $H^1(\mathbb{R} \times \Omega, e^{-x})$, one has $\Delta u = \partial_1 u$ and thus $\Delta u \in L^2(\mathbb{R} \times \Omega, e^{-x})$. Hence also

\begin{equation}
(2.25) \quad \xi \rightarrow \Delta u(\xi) \varphi(\xi - y) \in L^1(\mathbb{R} \times \Omega \setminus B_\varepsilon(y)).
\end{equation}

The fact that $u$ solves (2.17) together with (2.20) now implies

\[
\int_{\mathbb{R} \times \Omega \setminus B_\varepsilon(y)} u(\xi) (\Delta \varphi(\xi - y) + \partial_1 \varphi(\xi - y)) - \varphi(\xi - y) (\Delta u(\xi) - \partial_1 u(\xi)) \, d\xi = 0.
\]

From the integrability established in (2.22),(2.23),(2.24), and (2.25), it follows that

\begin{equation}
(2.26) \quad \int_{\mathbb{R} \times \Omega \setminus B_\varepsilon(y)} u(\xi) \Delta \varphi(\xi - y) - \varphi(\xi - y) \Delta u \, d\xi \\
+ \int_{\mathbb{R} \times \Omega \setminus B_\varepsilon(y)} u(\xi) \partial_1 \varphi(\xi - y) + \partial_1 u(\xi) \varphi(\xi - y) \, d\xi = 0.
\end{equation}
Now Green's Formula will be applied to the first integral above. However, since \( u \) is not necessarily in \( H_0^2(\mathbb{R} \times \Omega \setminus B_\varepsilon(y), e^{-\xi_1}) \), the integrability conditions for applying Green's Formula are not necessarily satisfied and we cannot apply it directly. Hence an approximation is made. For each \( N \in \mathbb{N} \) choose a function \( \chi_N \in C^\infty(\mathbb{R}) \) satisfying

\[
\chi_N = 1 \text{ on } (-N, N), \quad \chi_N = 0 \text{ on } \mathbb{R} \setminus (-N + 1, N + 1),
\]

\[
|\chi_N'| \leq 2, \quad \text{and} \quad |\chi_N''| \leq 2.
\]

The function \( (\xi_1, \xi_2, \xi_3) \rightarrow \chi_N(\xi_1)u(\xi_1, \xi_2, \xi_3) \) then satisfies

\[
\Delta [\chi_N u] = \chi_N \Delta u + 2\chi_N' \partial_1 u + \chi_N'' u,
\]

\[
\partial_1 [\chi_N u] = \chi_N' u + \chi_N \partial_1 u,
\]

\[
\frac{\partial}{\partial n} [\chi_N u] = \chi_N \frac{\partial u}{\partial n} \quad \text{on } \mathbb{R} \times \Gamma.
\]

We now replace \( u \) with \( \chi_N u \) in each integrand in (2.26). First

\[
\int_{\mathbb{R} \times \Omega \setminus B_\varepsilon(y)} \varphi(\xi - y) \Delta [\chi_N u] \, d\xi = \int_{\mathbb{R} \times \Omega \setminus B_\varepsilon(y)} \varphi(\xi - y) \chi_N \Delta u \, d\xi
\]

\[
+ \int_{\mathbb{R} \times \Omega \setminus B_\varepsilon(y)} \varphi(\xi - y) 2\chi_N' \partial_1 u \, d\xi
\]

\[
+ \int_{\mathbb{R} \times \Omega \setminus B_\varepsilon(y)} \varphi(\xi - y) \chi_N'' u \, d\xi
\]

\[
= I_1^N + I_2^N + I_3^N.
\]

From the integrability of \( \xi \rightarrow \varphi(\xi - y) \partial_1 u(\xi) \) and the Dominated Convergence Theorem it follows that

\[
|I_2^N| \leq \int_{\mathbb{R} \times \Omega \setminus B_\varepsilon(y)} 4 \chi\{-(N+1), -N\} \cup\{N, N+1\}(\xi_1) \left| \varphi(\xi - y) \partial_1 u \right| \, d\xi \rightarrow 0 \quad \text{for } N \rightarrow \infty.
\]

Here \( \chi[a, b] \) denotes the indicator function of the interval \([a, b]\). Similarly

\[
|I_3^N| \leq \int_{\mathbb{R} \times \Omega \setminus B_\varepsilon(y)} 4 \chi\{-(N+1), -N\} \cup\{N, N+1\}(\xi_1) \left| \varphi(\xi - y) u \right| \, d\xi \rightarrow 0 \quad \text{for } N \rightarrow \infty
\]

and

\[
|I_1^N| = \int_{\mathbb{R} \times \Omega \setminus B_\varepsilon(y)} \varphi(\xi - y) \Delta u \, d\xi \leq \int_{\mathbb{R} \times \Omega \setminus B_\varepsilon(y)} \chi\{\mathbb{R} \setminus (-N, N)\}(\xi_1) \left| \varphi(\xi - y) \Delta u \right| \, d\xi
\]

\[
\rightarrow 0 \quad \text{for } N \rightarrow \infty.
\]
Hence by (2.28)

\[
\int_{\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y)} \varphi(\xi - y) \Delta u \, d\xi = \lim_{N \to \infty} \int_{\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y)} \varphi(\xi - y) \Delta [\chi_N u] \, d\xi .
\]

(2.29)

Analogously one has

\[
\int_{\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y)} u \Delta \varphi(\xi - y) \, d\xi = \lim_{N \to \infty} \int_{\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y)} [\chi_N u] \Delta \varphi(\xi - y) \, d\xi .
\]

(2.30)

From (2.27) and the fact that \( \chi_N(\xi) = 1 \) in a neighborhood of \( \partial B_{\varepsilon}(y) \) for large \( N \), it follows that

\[
\int_{\partial(\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y))} \varphi(\xi - y) \frac{\partial}{\partial n} [\chi_N u](\xi) \, d\xi = \int_{\partial B_{\varepsilon}(y)} \varphi(\xi - y) \frac{\partial u}{\partial n}(\xi) \, dS(\xi) + \int_{\mathbb{R} \times \Gamma} \varphi(\xi - y) \chi_N(\xi) \frac{\partial u}{\partial n}(\xi) \, dS(\xi)
\]

(2.31)

\[
\to \int_{\partial(\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y))} \varphi(\xi - y) \frac{\partial u}{\partial n}(\xi) \, dS(\xi) \quad \text{for } N \to \infty.
\]

Similarly

\[
\int_{\partial(\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y))} [\chi_N u] \frac{\partial \varphi}{\partial n}(\xi - y) \, dS(\xi) \to \int_{\partial(\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y))} u \frac{\partial \varphi}{\partial n}(\xi - y) \, dS(\xi) \quad \text{for } N \to \infty.
\]

(2.32)

By assumption (2.18), one has \([\chi_N u] \in H_{2}^{2}(\mathbb{R} \times \Omega)\). Consider the space \( H_{2}^{2} \) with the underlying domain being a finite part of the cylinder \( \mathbb{R} \times \Omega \setminus B_{\varepsilon}(y) \) containing the support of \( \chi_N \). Obviously \( \varphi(\cdot - y) \) lies in this space. Hence Green’s Formula can be applied to \([\chi_N u]\) and \(\varphi(\cdot - y)\) yielding

\[
\int_{\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y)} [\chi_N u](\xi) \Delta \varphi(\xi - y) - \varphi(\xi - y) \Delta [\chi_N u](\xi) \, d\xi = \int_{\partial(\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y))} [\chi_N u](\xi) \frac{\partial \varphi}{\partial n}(\xi - y) - \varphi(\xi - y) \frac{\partial [\chi_N u]}{\partial n} \, dS(\xi) .
\]

Now letting \( N \to \infty \) in the equation above, (2.29), (2.30), (2.31), and (2.32) imply

\[
\int_{\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y)} u(\xi) \Delta \varphi(\xi - y) - \varphi(\xi - y) \Delta u(\xi) \, d\xi = \int_{\partial(\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y))} u(\xi) \frac{\partial \varphi}{\partial n}(\xi - y) - \varphi(\xi - y) \frac{\partial u}{\partial n} \, dS(\xi) .
\]

(2.33)
2.3 Representation formula and decay estimates

Equation (2.33) concerns the first integral in (2.26). Now consider the second integral in (2.26). A similar approximation as above yields

\[
\int_{\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y)} u(\xi) \partial_1 \varphi(\xi - y) + \partial_1 u(\xi) \varphi(\xi - y) \, d\xi
\]

\[
= \lim_{N \to \infty} \int_{\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y)} [\chi_N u](\xi) \partial_1 \varphi(\xi - y) + \partial_1 [\chi_N u](\xi) \varphi(\xi - y) \, d\xi .
\]

Let \( n = (n_1, n_2, n_3) \) denote the outward normal on \( \partial (\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y)) \). By partial integration of the right-hand side above we obtain

\[
\int_{\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y)} u(\xi) \partial_1 \varphi(\xi - y) + \partial_1 u(\xi) \varphi(\xi - y) \, d\xi
\]

\[
= \lim_{N \to \infty} \int_{\partial (\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y))} [\chi_N u](\xi) \varphi(\xi - y) n_1(\xi) \, dS(\xi)
\]

\[
= \int_{\partial (\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y))} u(\xi) \varphi(\xi - y) n_1(\xi) \, dS(\xi)
\]

\[
= \int_{\partial B_{\varepsilon}(y)} u(\xi) \varphi(\xi - y) n_1(\xi) \, dS(\xi) .
\]

(2.34)

Since the first component of the normal on \( \mathbb{R} \times \Gamma \) is zero, the last integral above reduces to an integral over \( \partial B_{\varepsilon}(y) \).

Inserting (2.33) and (2.34) into (2.26), it finally follows that

\[
(2.35) \quad \int_{\partial (\mathbb{R} \times \Omega \setminus B_{\varepsilon}(y))} u(\xi) \frac{\partial \varphi}{\partial n}(\xi - y) - \varphi(\xi - y) \frac{\partial u}{\partial n}(\xi) \, dS(\xi) + \int_{\partial B_{\varepsilon}(y)} u(\xi) \varphi(\xi - y) n_1(\xi) \, dS(\xi) = 0 .
\]

Having established the above identity, the representation formula can now be proved
in the usual manner. One has

\[
\int_{\partial B_\varepsilon(y)} u(\xi) \frac{\partial \varphi}{\partial n}(\xi - y) \, dS(\xi) = - \frac{1}{\varepsilon} \int_{\partial B_\varepsilon(y)} u(\xi) D\varphi(\xi - y) \cdot (\xi - y) \, dS(\xi)
\]

\[= \frac{1}{\varepsilon} \int_{\partial B_\varepsilon(y)} u(\xi) \left( \frac{1}{|\xi - y|} + \frac{1}{2} + \frac{\xi_1 - y_1}{2|\xi - y|} \right) e^{-\frac{1}{2}|\xi - y| - \frac{1}{2}(\xi_1 - y_1)} \, dS(\xi)
\]

\[= \int_{\partial B_\varepsilon(y)} u(\xi) \left( \frac{1}{\varepsilon^2} + \frac{1}{2\varepsilon} + \frac{\xi_1 - y_1}{2\varepsilon^2} \right) e^{-\frac{\varepsilon}{2} - \frac{1}{2}(\xi_1 - y_1)} \, dS(\xi) .
\]

(2.36)

As noted in the beginning of this section, the conditions in (2.18) imply by standard
regularity theory for elliptic equations that \( u \) is continuous. Hence

\[
\int_{\partial B_\varepsilon(y)} u(\xi) \frac{1}{\varepsilon^2} e^{-\frac{\varepsilon}{2} - \frac{1}{2}(\xi_1 - y_1)} \, dS(\xi) = 3 \omega_3 \int_{\partial B_\varepsilon(y)} u(\xi) e^{-\frac{\varepsilon}{2} - \frac{1}{2}(\xi_1 - y_1)} \, dS(\xi)
\]

\[\rightarrow 3 \omega_3 u(y) \quad \text{for} \quad \varepsilon \rightarrow 0 .
\]

Similarly

\[
\int_{\partial B_\varepsilon(y)} u(\xi) \frac{1}{2\varepsilon} e^{-\frac{\varepsilon}{2} - \frac{1}{2}(\xi_1 - y_1)} \, dS(\xi) = 3 \omega_3 \frac{\varepsilon}{2} \int_{\partial B_\varepsilon(y)} u(\xi) e^{-\frac{\varepsilon}{2} - \frac{1}{2}(\xi_1 - y_1)} \, dS(\xi)
\]

\[\rightarrow 0 \quad \text{for} \quad \varepsilon \rightarrow 0 ,
\]

and

\[
\int_{\partial B_\varepsilon(y)} \frac{\xi_1 - y_1}{2\varepsilon^2} e^{-\frac{\varepsilon}{2} - \frac{1}{2}(\xi_1 - y_1)} \, dS(\xi) = 3 \omega_3 \int_{\partial B_\varepsilon(y)} u(\xi) \frac{1}{2}(\xi_1 - y_1) e^{-\frac{\varepsilon}{2} - \frac{1}{2}(\xi_1 - y_1)} \, dS(\xi)
\]

\[\rightarrow 0 \quad \text{for} \quad \varepsilon \rightarrow 0 .
\]

It thus follows from (2.36) that

\[
(2.37) \quad \int_{\partial B_\varepsilon(y)} u(\xi) \frac{\partial \varphi}{\partial n}(\xi - y) \, dS(\xi) \rightarrow 3 \omega_3 u(y) \quad \text{for} \quad \varepsilon \rightarrow 0 .
\]
Since standard regularity theory for elliptic equations also implies continuity of $Du$, one has for the second integrand in (2.35) that

$$\int_{\partial B_{\varepsilon}(y)} \varphi(\xi - y) \frac{\partial u}{\partial n}(\xi) \, dS(\xi) = \int_{\partial B_{\varepsilon}(y)} \frac{1}{|y - \xi|} e^{-\frac{1}{2}|\xi - y| - \frac{1}{2}(\xi_1 - y_1)} \, Du(\xi) \cdot \frac{y - \xi}{|y - \xi|} \, dS(\xi) = 3\omega_3 \int_{\partial B_{\varepsilon}(y)} Du(\xi) \cdot (y - \xi) \, e^{-\frac{1}{2}|\xi - y| - \frac{1}{2}(\xi_1 - y_1)} \, dS(\xi) \to 0 \quad \text{for } \varepsilon \to 0 .$$

(2.38)

Finally also

$$\int_{\partial B_{\varepsilon}(y)} u(\xi) \varphi(\xi - y) n_1(\xi) \, dS(\xi) = \int_{\partial B_{\varepsilon}(y)} u(\xi) \frac{1}{|y - \xi|} e^{-\frac{1}{2}|\xi - y| - \frac{1}{2}(\xi_1 - y_1)} \, \frac{y - \xi_1}{|y - \xi|} \, dS(\xi) = 3\omega_3 \int_{\partial B_{\varepsilon}(y)} u(\xi) \, e^{-\frac{1}{2}|\xi - y| - \frac{1}{2}(\xi_1 - y_1)} \, (y_1 - \xi_1) \, dS(\xi) \to 0 \quad \text{for } \varepsilon \to 0 .$$

(2.39)

Now letting $\varepsilon \to 0$ in (2.35), it follows from (2.37), (2.38), and (2.39) that

$$3\omega_3 \, u(y) = \int_{\mathbb{R} \times \Gamma} \frac{\partial u}{\partial n}(\xi) \varphi(\xi - y) - u(\xi) \frac{\partial \varphi}{\partial n}(\xi - y) \, dS(\xi) .$$

Substituting $g$ for $\frac{\partial u}{\partial n}$ in the equation above completes the proof.  

Having established a representation formula, a pointwise decay estimate for solutions of (2.17) can now be obtained.

**Lemma 2.6.** Let $u$ be a solution of equation (2.17) satisfying (2.18). If

$$\int_{\mathbb{R} \times \Gamma} |\varphi(\xi - y) e^{\frac{1}{2}(\xi_1 - y_1)}| \, dS(\xi) \leq C_1 \quad \forall y \in \mathbb{R} \times \Gamma$$

(2.40)

and

$$\int_{\mathbb{R} \times \Gamma} \left| \frac{\partial \varphi}{\partial n}(\xi - y) e^{\frac{1}{2}(\xi_1 - y_1)} \right| \, dS(\xi) \leq C_2 \quad \forall y \in \mathbb{R} \times \Gamma$$

(2.41)

then

$$|u(y) e^{-\frac{\xi_1}{2}}| \leq C \left( \|g(\xi) e^{-\frac{\xi_1}{2}}\|_{L^4(\mathbb{R} \times \Gamma)} + \|u(\xi) e^{-\frac{\xi_1}{2}}\|_{L^4(\mathbb{R} \times \Gamma)} \right)$$

(2.42)

for all $y \in \mathbb{R} \times \Gamma$, with $C$ depending only on $\Gamma$. 

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2.3 Representation formula and decay estimates

Proof.

For any \( y \in \mathbb{R} \times \Omega \) we have by Theorem 2.5 the representation

\[
(2.43) \quad u(y) = \frac{1}{3\omega_3} \int_{\mathbb{R} \times \Gamma} g(\xi) \varphi(\xi - y) - u(\xi) \frac{\partial \varphi}{\partial n}(\xi - y) \, dS(\xi)
\]

of \( u \). Let \( y^0 \in \mathbb{R} \times \Gamma \). We now examine the limit behavior of the above equation as \( y \) tends to \( y^0 \).

By assumption, \( u \in H^2((-N,N) \times \Omega) \) for any \( N \in \mathbb{N} \). Thus the Sobolev Embedding Theorem implies \( u \in C^{0,\alpha}(\mathbb{R} \times \Omega) \) for any \( 0 \leq \alpha < 1 \). Hence \( u(y) \to u(y^0) \) for \( y \to y^0 \) follows by continuity.

Now consider the right-hand side of (2.43). One can identify the integral as a sum of a single and double layer potential with respect to the fundamental solution \( \varphi \). Since the singularity of \( \varphi \) is of the same order as the singularity of the Newtonian potential, the limit behavior of these potentials as \( y \) tends to \( y^0 \) is similar to that known from classical potential theory. This is precisely formulated in Theorem 14.I and Theorem 15.II in [Mir70]. More specifically, we have

\[
(2.44) \quad \frac{1}{3\omega_3} \int_{\mathbb{R} \times \Gamma} g(\xi) \varphi(\xi - y) - u(\xi) \frac{\partial \varphi}{\partial n}(\xi - y) \, dS(\xi) \to \quad \frac{1}{3\omega_3} \int_{\mathbb{R} \times \Gamma} g(\xi) \varphi(\xi - y^0) - u(\xi) \frac{\partial \varphi}{\partial n}(\xi - y^0) \, dS(\xi) + \frac{1}{2} u(y^0) \quad \text{for} \quad y \to y^0 .
\]

Since the book [Mir70] by Carlo Miranda does not contain any rigorous proofs of these theorems and many of the references are somewhat inaccessible, we shall briefly sketch a proof here.

Let \( E_3 \) denote the Newtonian potential

\[
E_3(y) := \frac{-1}{3\omega_3 |y|} \quad \forall y \in \mathbb{R}^3 \setminus \{0\} .
\]

By definition of \( \varphi \) we have

\[
(2.45) \quad \frac{1}{3\omega_3} \int_{\mathbb{R} \times \Gamma} g(\xi) \varphi(\xi - y) - u(\xi) \frac{\partial \varphi}{\partial n}(\xi - y) \, dS(\xi) =
\]

\[
- \int_{\mathbb{R} \times \Gamma} g(\xi) \, E_3(\xi - y) \, e^{-\frac{1}{2}|\xi-y| - \frac{1}{2}(|\xi| - |y|)} \, dS(\xi)
\]

\[
+ \int_{\mathbb{R} \times \Gamma} u(\xi) \, E_3(\xi - y) \frac{\partial}{\partial n} \left( e^{-\frac{1}{2}|\xi-y| - \frac{1}{2}(|\xi| - |y|)} \right) \, dS(\xi)
\]

\[
+ \int_{\mathbb{R} \times \Gamma} u(\xi) \, \frac{\partial E_3}{\partial n}(\xi - y) \, e^{-\frac{1}{2}|\xi-y| - \frac{1}{2}(|\xi| - |y|)} \, dS(\xi) .
\]
Hence we have the right-hand side of (2.43) expressed as a sum of single and double layer potentials of the Newtonian potential. Standard potential theory, see for example Proposition 10, §3, Chapter 2 in [DL00], implies continuity of the single layer potentials with respect to convergence towards a boundary point. Thus

\[
\int_{\mathbb{R} \times \Gamma} g(\xi) \text{E}_3(\xi - y) \, e^{-\frac{1}{2}|\xi - y| - \frac{1}{2}(\xi_1 - y_1)} \, dS(\xi) \to \\
\int_{\mathbb{R} \times \Gamma} g(\xi) \text{E}_3(\xi - y^0) \, e^{-\frac{1}{2}|\xi - y^0| - \frac{1}{2}(\xi_1 - y_1^0)} \, dS(\xi) \quad \text{for} \ y \to y^0
\]

and

\[
\int_{\mathbb{R} \times \Gamma} u(\xi) \text{E}_3(\xi - y) \, \frac{\partial}{\partial n} \left[ e^{-\frac{1}{2}|\xi - y| - \frac{1}{2}(\xi_1 - y_1)} \right] \, dS(\xi) \to \\
\int_{\mathbb{R} \times \Gamma} u(\xi) \text{E}_3(\xi - y^0) \, \frac{\partial}{\partial n} \left[ e^{-\frac{1}{2}|\xi - y^0| - \frac{1}{2}(\xi_1 - y_1^0)} \right] \, dS(\xi) \quad \text{for} \ y \to y^0 .
\]

Also for the double layer potential we can apply the standard potential theory. More precisely, mimicking the proof of Proposition 11, §3, Chapter 2 in [DL00] we obtain

\[
\int_{\mathbb{R} \times \Gamma} u(\xi) \frac{\partial}{\partial n} \left( \text{E}_3(\xi - y) \, e^{-\frac{1}{2}|\xi - y| - \frac{1}{2}(\xi_1 - y_1)} \right) \, dS(\xi) \to \\
\int_{\mathbb{R} \times \Gamma} u(\xi) \frac{\partial}{\partial n} \left( \text{E}_3(\xi - y^0) \, e^{-\frac{1}{2}|\xi - y^0| - \frac{1}{2}(\xi_1 - y_1^0)} \right) \, dS(\xi) + \frac{1}{2} u(y^0) \quad \text{for} \ y \to y^0 .
\]

Consequently, passing to the limit in (2.45) yields (2.44).

From (2.43), (2.44) and the continuity of \( u \) on the boundary, it now follows that

\[
u(y^0) = \frac{1}{3 \omega_3} \int_{\mathbb{R} \times \Gamma} g(\xi) \varphi(\xi - y^0) - u(\xi) \frac{\partial \varphi}{\partial n}(\xi - y^0) \, dS(\xi) + \frac{1}{2} u(y^0) .
\]

Hence

\[
\frac{1}{2} u(y^0) \, e^{-\frac{y_1^0}{2}} = \frac{1}{3 \omega_3} \int_{\mathbb{R} \times \Gamma} \left| g(\xi) \varphi(\xi - y^0) \, e^{-\frac{y_1^0}{2}} + u(\xi) \frac{\partial \varphi}{\partial n}(\xi - y^0) \, e^{-\frac{y_1^0}{2}} \right| \, dS(\xi) \leq \\
\frac{1}{3 \omega_3} \int_{\mathbb{R} \times \Gamma} \left| g(\xi) \, e^{-\frac{y_1^0}{2}} \varphi(\xi - y^0) \, e^{\frac{1}{2}(\xi_1 - y_1^0)} \right| \, dS(\xi) + \\
\frac{1}{3 \omega_3} \int_{\mathbb{R} \times \Gamma} \left| u(\xi) \, e^{-\frac{y_1^0}{2}} \frac{\partial \varphi}{\partial n}(\xi - y^0) \, e^{\frac{1}{2}(\xi_1 - y_1^0)} \right| \, dS(\xi) .
\]
Applying Hölder’s inequality thus yields

\[ \left| \frac{1}{2} u(y^0) e^{-\frac{\xi_1}{2}} \right| \leq \frac{1}{3 \omega_3} \| g(\xi) e^{-\frac{\xi_2}{2}} \|_{L^4(\mathbb{R} \times \Gamma)} \left( \int_{\mathbb{R} \times \Gamma} |\varphi(\xi - y^0) e^{\frac{1}{2} (\xi_1 - y^0)}| \frac{1}{2} dS(\xi) \right)^{\frac{3}{4}} + \frac{1}{3 \omega_3} \| u(\xi) e^{-\frac{\xi_2}{2}} \|_{L^4(\mathbb{R} \times \Gamma)} \left( \int_{\mathbb{R} \times \Gamma} \left| \frac{\partial \varphi}{\partial n}(\xi - y^0) e^{\frac{1}{2} (\xi_1 - y^0)} \right| \frac{1}{2} dS(\xi) \right)^{\frac{3}{4}}. \]

Finally, by assumption (2.40) and (2.41) it follows that

\[ |u(y^0) e^{-\frac{\xi_1}{2}}| \leq C \left( \| g(\xi) e^{-\frac{\xi_2}{2}} \|_{L^4(\mathbb{R} \times \Gamma)} + \| u(\xi) e^{-\frac{\xi_2}{2}} \|_{L^4(\mathbb{R} \times \Gamma)} \right) \]

with \( C \) depending only on \( \Gamma \).

Remark 2.7. We will later see that \( \| u(\xi) e^{-\frac{\xi_1}{2}} \|_{L^4(\mathbb{R} \times \Gamma)} < \infty \) holds for all elements \( u \) in \( H^2_1(\mathbb{R} \times \Omega, e^{-x}) \). Hence we only need \( \| g(\xi) e^{-\frac{\xi_2}{2}} \|_{L^4(\mathbb{R} \times \Gamma)} \) to be finite in order to use the lemma above. As a result, when studying solutions of the equation

\[
\begin{cases} 
\Delta u - c \partial_x u = 0 & \text{in } \mathbb{R} \times \Omega, \\
\frac{\partial u}{\partial n} = f(u) & \text{on } \mathbb{R} \times \partial \Omega
\end{cases}
\]

we only need suitable growth conditions on \( f \). If, for example, \( f \) has at most linear growth then \( \| f(u) e^{-\frac{\xi_1}{2}} \|_{L^4(\mathbb{R} \times \Gamma)} < \infty \) follows and we obtain an exponential decay estimate. In other words, we do not need \( f \) to have any particular shape or number of vanishing points. This later turns out to be one of the key reasons why we are able to handle a fairly large class of non-linearities \( f \).

Example 2.8. The unit-ball \( B_1(0) \), or more specifically its boundary \( \Gamma = \partial B_1(0) \), satisfies the conditions (2.40) and (2.41) in Lemma 2.6.

Fix \( y \in \mathbb{R} \times \Gamma \). For \( \xi \in \mathbb{R} \times \Gamma \) one has

\[ |\xi - y|^2 = (\xi_1 - y_1)^2 + 2 (1 - (\xi_1, y_1)) \left( y_1 - y_2 \right). \]
It follows that
\[
\int_{\mathbb{R} \times \Gamma} \left| \varphi(\xi - y) e^{\frac{i}{2} (\xi_1 - y_1)} \right|^{\frac{4}{3}} dS(\xi)
\]
\[
= \int_{\mathbb{R} \times \Gamma} \frac{1}{|\xi - y|^\frac{2}{3}} e^{-\frac{4}{3}|\xi - y|} dS(\xi)
\]
\[
= \int_{\mathbb{R} \times \Gamma} \frac{1}{((\xi_1 - y_1)^2 + 2(1 - (\xi_1 \cdot y_1)/(y_1)))^{\frac{2}{3}}} e^{-\frac{4}{3} \sqrt{((\xi_1 - y_1)^2 + 2(1 - (\xi_1 \cdot y_1)/(y_1)))}} dS(\xi)
\]
\[
\leq C_1 \int_{-1}^{1} \int_{\Gamma} \frac{1}{(\xi_1^2 + 2(1 - (\xi_2 \cdot y_3)/(y_3)))^{\frac{2}{3}}} dS(\xi_2, \xi_3) d\xi_1 + C_2
\]

with \(C_1\) and \(C_2\) not depending on \(y\). By the rotational symmetry of \(\Gamma = \partial B_1(0)\), it can be assumed without loss of generality that \((\xi_2 \cdot y_3)/(y_3) = (\xi_1 \cdot \eta)\). Let \(\partial B_1(0)^{++}\) denote the part of \(\partial B_1(0)\) lying in the upper positive half of \(\mathbb{R}^2\). Obviously
\[
\int_{-1}^{1} \int_{\Gamma} \frac{1}{(\xi_1^2 + 2(1 - (\xi_2 \cdot \eta)/(\eta)))^{\frac{2}{3}}} dS(\xi_2, \xi_3) d\xi_1
\]
\[
\leq C_1 \int_{0}^{1} \int_{\partial B_1(0)^{++}} \frac{1}{(\xi_1^2 + 2(1 - (\xi_2 \cdot \eta)/(\eta)))^{\frac{2}{3}}} dS(\xi_2, \xi_3) d\xi_1.
\]

Using the parametrisation
\[
\gamma(t) = (t, \sqrt{1 - t^2}), \quad t \in (0, 1)
\]
\[
dS(\xi_2, \xi_3) = \frac{1}{\sqrt{1 - t^2}} dt
\]
of $\partial B_1(0)^{++}$, it follows that

$$
\int_{-1}^{1} \int_{0}^{1} \frac{1}{(\xi_1^2 + 2(1 - (\xi_2)/(\xi_3)/0))^{\frac{3}{2}}} \mathrm{d}S(\xi_2, \xi_3) \mathrm{d}\xi_1 \\
\leq C \int_{0}^{1} \int_{0}^{1} \frac{1}{(\xi_1^2 + 2(1-t))^{\frac{3}{2}}} \frac{1}{\sqrt{1-t^2}} \mathrm{d}t \mathrm{d}\xi_1 \\
= C \int_{0}^{1} \int_{0}^{1} \frac{1}{(\xi_1^2 + 2(1-t))^{\frac{3}{2}}} \frac{1}{\sqrt{(1-t)(1+t)}} \mathrm{d}t \mathrm{d}\xi_1 \\
\leq C \int_{0}^{1} \int_{0}^{\sqrt{2}} \frac{1}{(\xi_1^2 + s^2)^{\frac{3}{2}}} \mathrm{d}s \mathrm{d}\xi_1 < \infty.
$$

(2.48)

The last integral being finite can easily be seen by switching to polar coordinates and integrating over a ball containing $[0, 1] \times [0, \sqrt{2}]$. By (2.47) and (2.48) one now has

$$
\int_{\mathbb{R} \times \Gamma} |\varphi(\xi - y) e^{\frac{1}{2}(\xi_1 - y_1)}|^{\frac{3}{2}} \mathrm{d}S(\xi) \leq C < \infty
$$

with $C$ not depending on $y$. Hence condition (2.40) is satisfied.

In order to show (2.41), the normal derivative of $\xi \to \varphi(\cdot - y)$ is calculated for $\xi \in \mathbb{R} \times \Gamma = \mathbb{R} \times \partial B_1(0)$. One has

$$
\frac{\partial \varphi}{\partial n}(\xi - y) = D\varphi(\xi - y) \cdot \begin{pmatrix} 0 \\ \xi_2 \\ \xi_3 \end{pmatrix} \\
= \left((\xi_2, \xi_3, \xi_4) - 1\right) \left(\frac{1 + \frac{1}{2}|\xi - y|}{|\xi - y|^3}\right) e^{-\frac{1}{2}|\xi - y| - \frac{1}{2}(\xi_1 - y_1)}.
$$

(2.49)
Using (2.46) now yields
\[
\int_{\mathbb{R} \times \Gamma} |\frac{\partial \varphi}{\partial n}(\xi - y)| \, e^{\frac{i}{2}(\xi_1 - y_1)} \left| \frac{\xi}{|\xi - y|} \right|^\frac{4}{3} \, dS(\xi)
\]
\[
= \int_{\mathbb{R} \times \Gamma} \frac{|(\frac{\xi_2}{\xi_3}) \cdot (\frac{y_2}{y_3}) - 1| \frac{4}{3}}{|\xi - y|^4} \left(1 + \frac{1}{2}|\xi - y|\right) \frac{4}{3} \, e^{-\frac{4}{5}|\xi - y|} \, dS(\xi)
\]
\[
= \int_{\mathbb{R} \times \Gamma} \frac{1}{((\xi_1 - y_1)^2 + 2(1 - (\frac{\xi_2}{\xi_3}) \cdot (\frac{y_2}{y_3}))^2} \left(1 + \frac{1}{2}\sqrt{((\xi_1 - y_1)^2 + 2(1 - (\frac{\xi_2}{\xi_3}) \cdot (\frac{y_2}{y_3}))} \right)^{\frac{4}{3}}
\]
\[
e^{-\frac{4}{5}\sqrt{(\xi_1 - y_1)^2 + 2(1 - (\frac{\xi_2}{\xi_3}) \cdot (\frac{y_2}{y_3}))}} \, dS(\xi)
\]
\[
\leq C_1 \int_{-1}^{1} \int_{\Gamma} \frac{1}{(\xi_1^2 + 2(1 - (\frac{\xi_2}{\xi_3}) \cdot (\frac{y_2}{y_3}))^2} \, dS(\xi_2, \xi_3) \, d\xi_1 + \frac{2}{\xi_1^4} \int_{1}^{\infty} \int_{\Gamma} \left(1 + \frac{1}{2}\sqrt{\xi_1^2 + 4}\right) \frac{4}{3} \, e^{-\frac{4}{5}\xi_1} \, dS(\xi_2, \xi_3) \, d\xi_1
\]
\[
\leq C_1 \int_{-1}^{1} \int_{\Gamma} \frac{1}{(\xi_1^2 + 2(1 - (\frac{\xi_2}{\xi_3}) \cdot (\frac{y_2}{y_3}))^2} \, dS(\xi_2, \xi_3) \, d\xi_1 + C_2
\]

with \( C_1 \) and \( C_2 \) not depending on \( y \). Once more due to the rotational symmetry of \( \Gamma = \partial B_1(0) \), it can be assumed without loss of generality that \((\frac{y_2}{y_3}) = (0)\). It follows that
\[
\int_{-1}^{1} \int_{\Gamma} \frac{1}{(\xi_1^2 + 2(1 - (\frac{\xi_2}{\xi_3}))^2} \, dS(\xi_2, \xi_3) \, d\xi_1
\]
\[
= \int_{-1}^{1} \int_{\Gamma} \frac{1}{(\xi_1^2 + 2(1 - \xi_2))^2} \, dS(\xi_2, \xi_3) \, d\xi_1
\]
\[
\leq C \int_{-1}^{1} \int_{\partial B_1(0)^+} \frac{1}{(\xi_1^2 + 2(1 - \xi_2))^2} \, dS(\xi_2, \xi_3) \, d\xi_1.
\]
Again using the parametrisation $\gamma$ of $\partial B_1(0)^{++}$ now yields

$$
\int_0^1 \int_{B_1(0)^{++}} \frac{(1-\xi_2)^{\frac{4}{3}}}{(\xi_1^2 + 2(1-\xi_2))^2} dS(\xi_2, \xi_3) d\xi_1
$$

$$
= \int_0^1 \int_0^{1} \frac{(1-t)^{\frac{4}{3}}}{(\xi_1^2 + 2(1-t))^2} \frac{1}{\sqrt{1-t^2}} \frac{1}{(1-t)^\frac{1}{2} (1+t)^\frac{1}{2}} dt d\xi_1
$$

$$
\leq C \int_0^1 \int_0^{1} \frac{(1-t)^{\frac{2}{3}}}{(\xi_1^2 + 2(1-t))^2} dt d\xi_1
$$

$$
= C \int_0^{\sqrt{2}} \frac{s^{\frac{4}{3}}}{(\xi_1^2 + s^2)^2} ds d\xi_1 < \infty .
$$

Switching to polar coordinates and integrating over a ball containing $[0, 1] \times [0, \sqrt{2}]$ shows that the last integral above is finite. Hence by (2.51) and (2.50) it follows that

(2.52) $$\int_{\mathbb{R} \times \Gamma} \left| \frac{\partial \varphi}{\partial n} (\xi - y) e^{\frac{1}{2}(\xi_1 - y_1)} \right|^{\frac{4}{3}} dS(\xi) \leq C < \infty$$

with $C$ not depending on $y$. Hence condition (2.41) is satisfied.

Remark 2.9. Since the conditions (2.40) and (2.41) in Lemma 2.6 holds for a ball, they are likely to hold for any domain which can be mapped sufficiently smooth to a ball.

2.4 Main theorem

Existence of a minimizer in problem (1.15) can now be proved. More precisely, we define

(2.53) $$\mathcal{E}(u) := \frac{1}{2} \int_{\mathbb{R} \times \Omega} |Du|^2 e^{-x} \, d(x,y) ,$$

(2.54) $$J(u) := \int_{\mathbb{R} \times \Gamma} F(u) e^{-x} \, dS(y) \, dx ,$$

(2.55) $$C := \{ u \in H_2^1(\mathbb{R} \times \Omega, e^{-x}) \mid J(u) = 1 \} ,$$
and consider the problem

\[ (2.56) \quad \mathcal{E} \to \operatorname{Min} \text{ in } \mathcal{C} \]

of minimizing \( \mathcal{E} \) over the class \( \mathcal{C} \). In Theorem 2.13 the existence of a minimizer hereof is established under suitable conditions on \( f \) and \( \Gamma \).

First, the results of the previous two sections are used to construct a minimizing sequence for (2.56). It is then shown that this sequence converges weakly to a minimizer of (2.56). Essential are the estimates in Lemma 2.6 as they turn out to compensate for the lack of compactness caused by the failing of \( H^1_2(\mathbb{R} \times \Omega, e^{-x}) \) to be compactly embedded into \( L^2(\mathbb{R} \times \Gamma, e^{-x}) \).

In order to construct the minimizing sequence, a sequence of approximating problems of the type encountered in Section 2.2 is considered. Define for \( n \in \mathbb{N} \) the functions

\[ (2.57) \quad \vartheta_n(x) := e^{-\frac{|x|}{n}}, \quad x \in \mathbb{R}. \]

Note that \( \vartheta_n \) satisfies condition (2.6). Furthermore define

\[ (2.58) \quad J_n(u) := \int_{\mathbb{R} \times \Gamma} \vartheta_n(x) F(u) e^{-x} \, dS(y) \, dx \]

and

\[ (2.59) \quad \mathcal{C}_n := \{ u \in H^1_2(\mathbb{R} \times \Omega, e^{-x}) \mid J_n(u) = 1 \}. \]

Assuming \( f \) satisfies (2.7), Theorem 2.2 ensures the existence of a minimizer \( u_n \) of the problem

\[ (2.60) \quad \mathcal{E} \to \operatorname{Min} \text{ in } \mathcal{C}_n. \]

The hereby induced sequence \( \{ u_n \}_{n=1}^{\infty} \) will serve as a basis for the minimizing sequence. In fact \( \{ u_n \}_{n=1}^{\infty} \) turns out to satisfy the minimizing property

\[ (2.61) \quad \mathcal{E}(u_n) \to \inf_{u \in \mathcal{C}} \mathcal{E}(u) \quad \text{for } n \to \infty. \]

Since the elements \( \{ u_n \}_{n=1}^{\infty} \) do not belong to the class \( \mathcal{C} \) of admissible elements of problem (2.56), the sequence itself is not a minimizing sequence of (2.56). However, admissibility with respect to problem (2.56) can be obtained by scaling the elements appropriately. As \( n \) tends to infinity, these scales tend to 1. Hence also the sequence of scaled elements satisfies the minimizing property (2.61) turning it into a proper minimizing sequence of (2.56). More precisely, we have the following theorem.
Theorem 2.10. Assume $f$ satisfies (2.7). If $\{u_n\}_{n=1}^{\infty}$ is a sequence with each element a minimizer of problem (2.60), that is

$$\mathcal{E}(u_n) = \inf_{u \in C_n} \mathcal{E}(u) \quad \text{and} \quad u_n \in C_n,$$

then

$$\mathcal{E}(u_n) \to \inf_{u \in C} \mathcal{E}(u) \quad \text{for } n \to \infty.$$

Furthermore there exists a sequence of real numbers $\{s_n\}_{n=1}^{\infty}$ such that

1. $\{s_n u_n\}_{n=1}^{\infty}$ is a minimizing sequence for $\mathcal{E}$ over $C$,
2. $0 < s_n \leq 1$ for all $n \in \mathbb{N}$, and
3. $s_n \to 1$ for $n \to \infty$.

Proof.

Put $I = \inf_{u \in C} \mathcal{E}(u)$. Since $f$ satisfies (2.7), it follows that

$$|F(u)| \leq \int_{0}^{u} |f(t)| dt \leq \int_{0}^{u} k \, t \, dt = \frac{1}{2} k \, u^2.$$ 

Hence for $u \in C$ one has

$$1 = J(u) = \int_{\mathbb{R} \times \Gamma} F(u) \, e^{-x} \, dS(y) \, dx \leq \frac{1}{2} k \int_{\mathbb{R} \times \Gamma} u^2 \, e^{-x} \, dS(y) \, dx.$$

Using the boundedness of the trace-operator $T : H^1_0(\mathbb{R} \times \Omega, e^{-x}) \to L^2(\mathbb{R} \times \Gamma, e^{-x})$ and Lemma 2.1, it follows that

$$1 \leq C \int_{\mathbb{R} \times \Omega} (u^2 + |D\mathbf{u}|^2) \, e^{-x} \, dS(y) \, dx \leq 10 \, C \, \mathcal{E}(u).$$

Consequently $I > 0$.

Let $\varepsilon > 0$ be given. Now choose $v \in C$ with $\mathcal{E}(v) < I + \varepsilon$. Consider for $h \in \mathbb{R}$ the translation $\tau_h v$ of $v$ by $h$ in the $x$-variable,

$$\tau_h v(x, y) := v(x + h, y).$$
For $h > 0$ and $n \geq 2$ it holds that

$$
\vartheta_n(x-h) = e^{-\frac{|x-h|}{n}} = \begin{cases} 
\frac{x-h}{n} & \text{for } x \leq h \\
\frac{h-x}{n} & \text{for } h < x
\end{cases}
$$

By condition (2.7), $f$ satisfies $0 \leq f(t) t$. Consequently $F \geq 0$. Hence

$$
\begin{align*}
J_n(\tau_h v) &= \int_{\mathbb{R} \times \Gamma} \vartheta_n(x) \, F(v(x+h,y)) \, e^{-x} \, dS(y) \, dx \\
&= \int_{\mathbb{R} \times \Gamma} \vartheta_n(x-h) \, F(v(x,y)) \, e^{-(x-h)} \, dS(y) \, dx \\
&\geq e^{-\frac{h}{n}} \int_{\mathbb{R} \times \Gamma} \vartheta_n(x) \, F(v(x,y)) \, e^{h} \, e^{-x} \, dS(y) \, dx \\
&= e^{\frac{h}{n}} J_n(v) .
\end{align*}
$$

Since $0 < \vartheta_n \leq 1$ and $F$ is non-negative, it follows that $0 < J_n(v) \leq 1$. Thus by (2.64) one can choose a sufficiently large $h$ such that $J_n(\tau_h v) = 1$. For each $n \in \mathbb{N}$ choose such a $h$ and denote it $h_n$. One then has

$$
|1 - J_n(v)| = J_n(\tau_{h_n} v) - J_n(v) \geq (e^{\frac{h_n}{2}} - 1) J_n(v) .
$$

Since $\vartheta_n(x) \to 1$ pointwise for $n \to \infty$ and $F$ is non-negative, by the Dominated Convergence Theorem

$$
J_n(v) = \int_{\mathbb{R} \times \Gamma} \vartheta_n(x) \, F(v) \, e^{-x} \, dS(y) \, dx \\
\to \int_{\mathbb{R} \times \Gamma} F(v) \, e^{-x} \, dS(y) \, dx = J(v) = 1 \quad \text{for } n \to \infty .
$$
Hence from (2.65) it follows that \((e^{h_n} - 1) \to 0\) for \(n \to \infty\). Consequently \(h_n \to 0\) and thus
\[
(e^{h_n} - 1) \leq \frac{\varepsilon}{I}
\]
for \(n\) sufficiently large.

Since \(J_n(\tau_{h_n} v) = 1\) one has \(\tau_{h_n} v \in C_n\). By assumption (2.62), \(u_n\) is a minimizer for \(E\) over \(C_n\). Hence
\[
E(u_n) \leq E(\tau_{h_n} v) = E(v) + E(\tau_{h_n} v) - E(v)
\]
(2.66)
\[
= E(v) + (e^{h_n} - 1) E(v)
\]
\[
\leq I + \varepsilon + \varepsilon = I + 2\varepsilon
\]
for \(n\) sufficiently large.

The minimizing property (2.63) now follows from (2.66) once it can be shown that \(I \leq E(u_n)\). Since \(F \geq 0\) and \(0 < \vartheta_n \leq 1\), one has \(1 = J_n(u_n) \leq J(u_n)\). Consider now the scaling \(s u_n\) of \(u_n\) by \(s \in \mathbb{R}^+\). By the Dominated Convergence Theorem it follows that
\[
J(s u_n) = \int_{\mathbb{R} \times \Gamma} F(s u_n) e^{-x} dS(y) dx \to 0 \quad \text{for} \quad s \to 0 .
\]
Thus there exists \(s \in \mathbb{R}\) with \(0 < s \leq 1\) such that \(J(s u_n) = 1\). Choosing \(s_n \in \mathbb{R}^+\) with this property implies \(s_n u_n \in C\) and consequently
\[
I \leq E(s_n u_n) = s_n^2 E(u_n) \leq E(u_n) .
\]
(2.67)
Thus we have established (2.63).

By (2.67) and (2.63) and the fact that \(s_n u_n \in C\), it is clear that \(\{s_n u_n\}_{n=1}^{\infty}\) is a minimizing sequence for \(E\) over \(C\) and \(s_n \to 1\). This completes the proof.

Having established the existence of a minimizing sequence \(\{s_n u_n\}_{n=1}^{\infty}\) for (2.56), consisting of scalings of minimizers \(u_n\) of (2.60), focus will now be on the weak limit hereof. By the same argument as in the proof of Theorem 2.2, the sequence \(\{s_n u_n\}_{n=1}^{\infty}\) is bounded in \(H^1_2(\mathbb{R} \times \Omega, e^{-x})\). Hence at least a subsequence converges weakly. It is now shown that this weak limit is a minimizer for (2.56). In order to do so, the following uniform pointwise bound on \(\{u_n\}\) is needed.

**Lemma 2.11.** Assume \(\Gamma\) satisfies (2.40) and (2.41). Further assume that \(f\) satisfies (2.7) and (2.14). Let \(\{u_n\}_{n=1}^{\infty}\) be a sequence in \(H^1_2(\mathbb{R} \times \Omega, e^{-x})\) satisfying (2.62). Then there exists an upper bound \(M\) such that
\[
|u_n(x, y) e^{-\frac{x}{2}}| \leq M \quad \forall (x, y) \in \mathbb{R} \times \Gamma
\]
(2.68)
for all \(n \in \mathbb{N}\).
Proof.

By assumption, \( u_n \) is a minimizer for \( \mathcal{E} \) over \( \mathcal{C}_n \). Thus \( u_n \) satisfies the corresponding Euler-Lagrange equation

\[
\int_{\mathbb{R} \times \Omega} Du \cdot Dv \ e^{-x} \ d(x,y) = \lambda_n \int_{\mathbb{R} \times \Gamma} \partial_n (u) \ f(u) \ v \ e^{-x} \ dS(y) \ dx
\]

for all \( v \in H^1_2(\mathbb{R} \times \Omega, e^{-x}) \). From Lemma 2.3 one has the bound

\[
0 < \lambda_n \leq \frac{2}{\Theta} \mathcal{E}(u_n)
\]

on the Lagrange multiplier \( \lambda_n \). By Theorem 2.10, \( \{u_n\}_{n=1}^\infty \) is a minimizing sequence for \( \mathcal{E} \) over \( \mathcal{C} \). Hence \( \{\mathcal{E}(u_n)\}_{n=1}^\infty \) is bounded and it follows from (2.70) that also \( \{\lambda_n\}_{n=1}^\infty \) is bounded by some constant \( L \).

Consider again the Euler-Lagrange equation (2.69). The fact that \( u_n \) satisfies (2.69) implies by Theorem 3.8 that \( u_n \in H^2_2((-N,N) \times \Omega, e^{-x}) \) for arbitrary \( N \in \mathbb{N} \). Hence the normal derivative of \( u_n \) exists on \( \mathbb{R} \times \Gamma \) at least in the trace sense. Applying the Fundamental Lemma of the Calculus of Variations thus yields

\[
\begin{aligned}
\Delta u_n - \partial_x u_n &= 0 \quad \text{in } \mathbb{R} \times \Omega, \\
\frac{\partial u_n}{\partial n} &= \lambda_n \partial_n f(u_n) \quad \text{on } \mathbb{R} \times \Gamma.
\end{aligned}
\]

It now follows from Lemma 2.6 that

\[
|u_n(x, y) e^{-\frac{x}{2}}| \leq C \left( \|\lambda_n \partial_n f(u_n) e^{-\frac{x}{2}}\|_{L^1(\mathbb{R} \times \Gamma)} + \|u_n e^{-\frac{x}{2}}\|_{L^4(\mathbb{R} \times \Gamma)} \right)
\]

\[
\leq C \left( L \|f(u_n) e^{-\frac{x}{2}}\|_{L^4(\mathbb{R} \times \Gamma)} + \|u_n e^{-\frac{x}{2}}\|_{L^4(\mathbb{R} \times \Gamma)} \right)
\]

for all \( (x, y) \in \mathbb{R} \times \Gamma \). The growth conditions imposed on \( f \) imply

\[
\|f(u_n) e^{-\frac{x}{2}}\|_{L^4(\mathbb{R} \times \Gamma)} \leq C \|u_n e^{-\frac{x}{2}}\|_{L^4(\mathbb{R} \times \Gamma)}.
\]

Thus

\[
(2.71) \quad |u_n(x, y) e^{-\frac{x}{2}}| \leq C \|u_n e^{-\frac{x}{2}}\|_{L^4(\mathbb{R} \times \Gamma)} \quad \forall (x, y) \in \mathbb{R} \times \Gamma.
\]

Now consider the Sobolev Embedding of \( H^1_2(\mathbb{R} \times \Omega) \) into \( L^p(\mathbb{R} \times \Gamma) \). The critical exponent of this embedding is 4. Note that the Sobolev Embedding does not hold for arbitrary unbounded domains. However, for a cylinder of the type \( \mathbb{R} \times \Omega \) used here there exists a continuous extension operator from the space \( H^1_2(\mathbb{R} \times \Omega) \) into \( H^1_2(\mathbb{R}^3) \) (See Theorem A.2). This means that \( \mathbb{R} \times \Omega \) is a so-called extension domain. Furthermore, \( \mathbb{R} \times \Omega \) satisfies the uniform \( C^1 \)-regularity condition (See Theorem A.1). For such domains the embedding holds as in the case of bounded domains (see for example Theorem 5.22 in [Ada75]). Hence

\[
\|v\|_{L^4(\mathbb{R} \times \Gamma)} \leq C \|v\|_{H^1_2(\mathbb{R} \times \Omega)} \quad \forall v \in H^1_2(\mathbb{R} \times \Omega).
\]
Thus by (2.71)
\[
|u_n e^{-\frac{t}{2s_n}}| \leq C \|u_n e^{-\frac{t}{2s_n}}\|_{H^2_0(\mathbb{R} \times \Omega)} \leq C \|u_n\|_{H^2_0(\mathbb{R} \times \Omega, e^{-x})} \leq C \mathcal{E}(u_n)
\]
for all \((x, y) \in \mathbb{R} \times \Gamma\). The last inequality is obtained using Lemma 2.1 in the usual way. The fact that \(\mathcal{E}(u_n)\) is bounded finally implies
\[
|u_n e^{-\frac{t}{2s_n}}| \leq M \forall (x, y) \in \mathbb{R} \times \Gamma
\]
for all \(n \in \mathbb{N}\).

Remark 2.12. The exponential decay estimate in the previous lemma is similar to the decay estimates established in [BN92], [BLL90], [Veg93], and [Hei88]. The methods we have used to obtain them is different though. While Lemma 2.11 is based on the potential theoretical arguments from Section 2.3, the decay estimates in [BN92], [BLL90], [Veg93], and [Hei88] are all obtained using maximum principles and comparison arguments. As mentioned in the introduction, the boundary condition \(\frac{\partial u}{\partial n} = f(u)\) complicates the use of maximum principles. Thus the potential theoretical approach seems to be better in this case. Furthermore, since it only calls for growth conditions on \(f\) to be imposed, it allows us to handle non-linearities vanishing only at 0. In other words, we avoid having to impose the condition \(f(0) = f(1) = 0\) which is essential in [BN92], [BLL90], and [Hei88].

The existence of a minimizer for problem (2.56) can now be proved. The minimizer is found as the weak limit of the minimizing sequence \(\{s_n u_n\}_{n=1}^\infty\) from Theorem 2.10. Essential to the proof is the pointwise bound from Lemma 2.11 which delivers sufficient information in order to control the term \(J(u_n)\) as \(n\) tends to infinity. Under suitable growth conditions on \(f\) this is adequate to ensure admissibility of the weak limit.

**Theorem 2.13.** Assume \(\Gamma\) satisfies (2.40) and (2.41). Furthermore assume \(f\) satisfies (2.7), (2.14), and
\[
\exists 0 < \alpha < 1, \ A > 0 : \ |f(t)| \leq A |t|^{\alpha} \text{ for all } t \in \mathbb{R} , \tag{2.72}
\]
\[
\exists \delta > 0, \ \beta > 1, \ B > 0 : \ |f(t)| \leq B |t|^\beta \text{ for all } |t| \leq \delta. \tag{2.73}
\]
Then there exists a minimizer for \(\mathcal{E}\) over \(\mathcal{C}\).

**Proof.**

By Theorem 2.2 there exists a minimizer \(u_n\) for \(\mathcal{E}\) over \(\mathcal{C}_n\). Furthermore, by Theorem 2.10 one can find a sequence of real numbers \(\{s_n\}_{n=1}^\infty\) with \(0 < s_n \leq 1\) and \(s_n \to 1\) such that \(\{s_n u_n\}_{n=1}^\infty\) is a minimizing sequence for \(\mathcal{E}\) over \(\mathcal{C}\). By Lemma 2.1, it follows that \(\{s_n u_n\}_{n=1}^\infty\) and thereby also \(\{u_n\}_{n=1}^\infty\) is bounded in \(H^2_0(\mathbb{R} \times \Omega, e^{-x})\). Hence a subsequence
of \( \{u_n\}_{n=1}^{\infty} \), which for the sake of simplicity will still be denoted \( \{u_n\}_{n=1}^{\infty} \), will converge weakly towards a function \( u \in H^1_0(\mathbb{R} \times \Omega, e^{-x}) \). The weak lower semicontinuity of \( \mathcal{E} \) implies \( \mathcal{E}(u) \leq \inf_{v \in \mathcal{C}} \mathcal{E}(v) \). It will now be shown that \( J(u) \geq 1 \), from which it easily follows that \( u \) is a proper minimizer.

Let \( \varepsilon > 0 \) be given. From Lemma 2.11 one has the pointwise bound

\[
|u_n(x, y) e^{-\frac{x}{L}}| \leq M \quad \forall (x, y) \in \mathbb{R} \times \Gamma
\]

uniformly in \( n \in \mathbb{N} \). Now choose \( L > 0 \) sufficiently large such that

\[
A |\Gamma| M^{\alpha+1} \frac{1}{(\alpha+1)(1-\frac{\alpha+1}{2})} e^{-(1-\frac{\alpha+1}{2})L} < \varepsilon,
\]

\[
B |\Gamma| M^{\beta+1} \frac{1}{(\beta+1)(\frac{\beta+1}{2}-1)} e^{-(\frac{\beta+1}{2}-1)L} < \varepsilon,
\]

and

\[
M e^{-\frac{L}{2}} < \delta.
\]

By assumption (2.72), it follows that

\[
\int_{L}^{\infty} \int_{\Gamma} F(s_n u_n) e^{-x} \, dS(y) \, dx \leq \int_{L}^{\infty} \int_{\Gamma} \frac{A}{1+\alpha} |s_n u_n|^{1+\alpha} e^{-x} \, dS(y) \, dx.
\]

Thus the bound from (2.74) implies

\[
\int_{L}^{\infty} \int_{\Gamma} F(s_n u_n) e^{-x} \, dS(y) \, dx \leq \int_{L}^{\infty} \int_{\Gamma} \frac{A}{1+\alpha} s_n^{1+\alpha} (M e^{\frac{x}{2}})^{1+\alpha} e^{-x} \, dS(y) \, dx
\]

\[
\leq \frac{A}{1+\alpha} |\Gamma| M^{1+\alpha} \int_{L}^{\infty} e^{-(1-\frac{1+\alpha}{2})x} \, dx
\]

\[
= \frac{A}{1+\alpha} |\Gamma| M^{1+\alpha} \frac{1}{(1-\frac{1+\alpha}{2})} e^{-(1-\frac{1+\alpha}{2})L}
\]

\[
< \varepsilon
\]

uniformly in \( n \in \mathbb{N} \).

From (2.74) it further follows that

\[
|s_n u_n| \leq s_n M e^{\frac{x}{2}} \leq M e^{-\frac{x}{2}} \quad \text{for } x < -L.
\]

Hence by (2.77) and assumption (2.73) one has

\[
\int_{-\infty}^{-L} \int_{\Gamma} F(s_n u_n) e^{-x} \, dS(y) \, dx \leq \int_{-\infty}^{-L} \int_{\Gamma} \frac{B}{1+\beta} |s_n u_n|^{\beta+1} e^{-x} \, dS(y) \, dx.
\]
Again using the bound from (2.74) yields
\begin{align}
\int_{-\infty}^{-L} \int_{\Gamma} F(s_n u_n) \ e^{-x} \ dS(y) dx & \leq \int_{-\infty}^{-L} \int_{\Gamma} B \ \frac{s_n^{1+\beta}}{1+\beta} (M e^{\frac{\beta}{2}})^{\beta+1} \ e^{-x} \ dS(y) dx \\
& \leq \frac{B}{1+\beta} \ |\Gamma| \ M^{\beta+1} \ \int_{-\infty}^{-L} e^{(\frac{\beta+1}{2}-1)x} dx \\
& = \frac{B}{1+\beta} \ |\Gamma| \ M^{\beta+1} \ \frac{1}{(\frac{\beta+1}{2}-1)} e^{-(\frac{\beta+1}{2}-1)L} \\
& < \varepsilon
\end{align}
(2.79)

uniformly in \( n \in \mathbb{N} \).

As in the proof of Theorem 2.2 one has
\begin{align}
\int_{-L}^{L} \int_{\Gamma} |F(u) - F(s_n u_n)| \ e^{-x} \ dS(y) dx & \leq C \int_{-L}^{L} |u - s_n u_n|^2 \ e^{-x} \ dS(y) dx \\
\leq 2 \varepsilon + \int_{-L}^{L} \int_{\Gamma} F(s_n u_n) \ e^{-x} \ dS(y) dx
\end{align}
(2.80)

Now consider the operator \( S_L \) constructed in the proof of Theorem 2.2. \( S_L : H^1_2(\mathbb{R} \times \Omega, e^{-x}) \to L^2((-L, L) \times \Gamma) \) is simply a composition of trace- and restriction-operators. As noted in the proof of Theorem 2.2, \( S_L \) is compact. Since \( u_n \to u \) in \( H^1_2(\mathbb{R} \times \Omega, e^{-x}) \), applying \( S_L \) to \( \{u_n\}_{n=1}^\infty \) thus implies \( u_n \to u \) strongly in \( L^2((-L, L) \times \Gamma) \). It follows that also \( s_n u_n \to u \) strongly in \( L^2((-L, L) \times \Gamma) \). Hence by (2.80) one has
\begin{align}
\int_{-L}^{L} \int_{\Gamma} F(s_n u_n) \ e^{-x} \ dS(y) dx & \to \int_{-L}^{L} \int_{\Gamma} F(u) \ e^{-x} \ dS(y) dx \\
\text{for } n \to \infty.
\end{align}
(2.81)

By (2.78), (2.79), and the fact that \( s_n u_n \in C \), it follows that
\begin{align}
1 = J(s_n u_n) & = \int_{\mathbb{R} \times \Gamma} F(s_n u_n) \ e^{-x} \ dS(y) dx \\
& < 2 \varepsilon + \int_{-L}^{L} \int_{\Gamma} F(s_n u_n) \ e^{-x} \ dS(y) dx
\end{align}

Letting \( n \to \infty \) implies by (2.81)
\begin{align}
1 - 2 \varepsilon & \leq \int_{-L}^{L} \int_{\Gamma} F(u) \ e^{-x} \ dS(y) dx \\
& \leq \int_{\mathbb{R} \times \Gamma} F(u) \ e^{-x} \ dS(y) dx = J(u).
\end{align}
The last inequality above holds since $F \geq 0$. Finally, letting $\varepsilon \to 0$ in the above yields $1 \leq J(u)$.

As in the proof of Theorem 2.2, one can now find an $s \in \mathbb{R}^+$ with $0 < s \leq 1$ such that $J(su) = 1$. It follows that $su \in \mathcal{C}$ and

$$
\mathcal{E}(su) = s^2 \mathcal{E}(u) \leq \mathcal{E}(u) \leq \inf_{v \in \mathcal{C}} \mathcal{E}(v) \leq \mathcal{E}(su).
$$

Consequently, $s = 1$ and $u$ is a minimizer for $\mathcal{E}$ over $\mathcal{C}$.

## 2.5 Asymptotics

We now turn our attention to the asymptotic behavior at $\pm \infty$ of the solution found in the previous section. From the representation formula in Theorem 2.5 we obtain, using the Hölder inequality as in the proof of Lemma 2.6, the estimate

$$
|u(x, y)| \leq M(y) e^{\frac{1}{2} x} \quad \forall x \in \mathbb{R}
$$

for any minimizer $u \in H^1_0(\mathbb{R} \times \Omega, e^{-x})$ of problem (2.56). Hence $u(x, y) \to 0$ as $x \to -\infty$ follows as an immediate consequence. Assuming $\Gamma$ and $f$ satisfy the conditions in Lemma 2.11, we can even prove as in Lemma 2.11 that

$$
|u(x, y)| \leq M e^{\frac{1}{2} x} \quad \forall (x, y) \in \mathbb{R} \times \Gamma.
$$

Hence also the boundary values of $u$ vanish at $-\infty$. In fact, since both decay estimates (2.82) and (2.83) hold for any solution of the associated Euler-Lagrange equation of (2.56), any such solution vanishes at $-\infty$. The asymptotic behavior at $+\infty$, however, is a more complicated matter.

The asymptotic behavior at $+\infty$ determines the type of travelling wave represented by $u$. If $u$ tends to 0 then we are dealing with a solitary wave. If on the other hand $u$ tends to some positive limit $v$ or infinity then $u$ represents a travelling front solution. As mentioned in the introduction, travelling front solutions are the interesting ones from a physical point of view. Hence it is desirable to at least rule out that the solution found in the previous section is a solitary wave.

When trying to determine the asymptotic behavior at $+\infty$, a drawback of working in the space $H^1_0(\mathbb{R} \times \Omega, e^{-x})$ becomes apparent. More specifically, most pointwise information which can be obtained on the solution comes in weighted form with $e^{-x}$ as weight. The estimates (2.82) and (2.83) are good examples of this. Consequently, one typically gains precise information at $-\infty$ and very little at $+\infty$. Nevertheless, we can prove that a solution $u$ does not vanish at $+\infty$ and thereby rule out that $u$ is a solitary wave. We can do so not only for minimizers of problem (2.56), but for any solution of the corresponding Euler-Lagrange equation.
Theorem 2.14. Assume \( f \) satisfies (2.7) and \( f \in C^2(\mathbb{R}) \) with \( f'' \) bounded. If \( u \in H^1_2(\mathbb{R} \times \Omega, e^{-x}) \) is a non-trivial solution of

\[
(2.84) \quad \int_{\mathbb{R} \times \Omega} D_u \cdot D_v \, e^{-x} \, d(x,y) = \lambda \int_{\mathbb{R} \times \Gamma} f(u) \, v \, e^{-x} \, dS(y) dx \quad \forall v \in H^1_2(\mathbb{R} \times \Omega, e^{-x})
\]

with \( \lambda > 0 \) then \( x \to \|u(x, \cdot)\|_{L^2(\Gamma)} \) does not vanish as \( x \) tends to \(+\infty\).

Proof.

Define

\[
(2.85) \quad \varphi(x) := \frac{1}{2} \int_{\Omega} |D_y u|^2 \, dy - \lambda \int_{\Gamma} F(u) \, dS(y) - \frac{1}{2} \int_{\Omega} (\partial_x u)^2 \, dy .
\]

By Theorem 3.8, \( u \in H^3_{2,B-loc}(\Omega) \). From the Sobolev Embedding Theorem it follows that \( u \in C^1(\mathbb{R} \times \overline{\Omega}) \). Furthermore, standard regularity theory for elliptic equations implies \( u \in C^\infty(\mathbb{R} \times \Omega) \). Hence we can differentiate \( \varphi \). We have

\[
\varphi'(x) = \int_{\Omega} D_y u \cdot D_y[\partial_x u] \, dy - \lambda \int_{\Gamma} f(u) \, \partial_x u \, dS(y) - \int_{\Omega} \partial_x u \, \partial_x^2 u \, dy .
\]

The regularity of \( u \) and (2.84) implies \( \frac{\partial u}{\partial n} = \lambda f(u) \). Thus by partial integration

\[
\varphi'(x) = -\int_{\Omega} \Delta_y u \, \partial_x u \, dy - \int_{\Omega} \partial_x u \, \partial_x^2 u \, dy .
\]

Additionally, (2.84) and the regularity of \( u \) implies \( \Delta_{(x,y)} u = \partial_x u \) in \( \mathbb{R} \times \Omega \). Hence

\[
\varphi'(x) = -\int_{\Omega} (\partial_x u - \partial_x^2 u) \, \partial_x u \, dy - \int_{\Omega} \partial_x u \, \partial_x^2 u \, dy
\]

\[
= -\int_{\Omega} (\partial_x u)^2 \, dy \leq 0 .
\]

Since \( u \) is non-trivial, \( \int_{\Omega} (\partial_x u)^2 \, dy \neq 0 \) for some \( x \in \mathbb{R} \). Consequently

\[
\alpha := \lim_{x \to -\infty} \varphi > \lim_{x \to \infty} \varphi := \beta .
\]

By the monotonicity of \( \varphi \) these limits exist. Since \( u \in H^1_2(\mathbb{R} \times \Omega, e^{-x}) \) we have

\[
(2.87) \quad -\infty < \int_{\mathbb{R}} \varphi(x) \, e^{-x} \, dx < \infty .
\]
This implies $\alpha = 0$ and hence $\beta < 0$. We deduce that $\varphi$ is non-positive.

Now assume $x \to \|u(x, \cdot)\|_{L^2(\Gamma)}$ vanishes at $+\infty$. Put

$$h(x) := \varphi'(x) - 2\varphi(x), \ x \in \mathbb{R}.$$  \hfill (2.88)

By (2.85) and (2.86) we have

$$h(x) = -\int_\Omega |D_y u|^2 \, dy + 2\lambda \int_\Gamma F(u) \, dS(y).$$

Since

$$\left| \int_\Gamma F(u) \, dS(y) \right| \leq C \int_\Gamma u^2 \, dS(y) \to 0 \ \text{for} \ x \to +\infty$$

it follows that $\limsup_{x \to +\infty} h \leq 0$. Now solving (2.88) with respect to $\varphi$, we obtain for any $t > t_0$

the representation

$$\varphi(t) = e^{2(t-t_0)} \left( \varphi(t_0) + \int_{t_0}^t h(x) \, e^{-2x} \, dx \right).$$ \hfill (2.89)

Since $\limsup_{x \to +\infty} h \leq 0$ we have for $t_0$ sufficiently large that

$$\int_{t_0}^t h(x) \, e^{-2x} \, dx \leq \frac{1}{2} e^{-2t_0} \ \forall \ t > t_0.$$  

Hence choosing $t_0$ sufficiently large such that $\frac{1}{2} e^{-2t_0} \leq -\frac{1}{4} \beta$ and $\varphi(t_0) \leq \frac{1}{2} \beta$ we obtain

$$\varphi(t_0) + \int_{t_0}^t h(x) \, e^{-2x} \, dx \leq \frac{1}{4} \beta \ \forall \ t > t_0.$$  

Then by (2.89)

$$\varphi(t) \leq e^{2(t-t_0)} \frac{1}{4} \beta \ \forall \ t > t_0$$

follows and consequently

$$\int_{t_0}^\infty \varphi(t) \, e^{-t} \, dt = -\infty.$$  

This contradicts (2.87). We conclude that $x \to \|u(x, \cdot)\|_{L^2(\Gamma)}$ does not vanish at $+\infty$. \hfill \Box

\textit{Remark 2.15.} By the theorem above and the boundedness of the trace operator $T : H^1_\delta(\Omega) \to L^2(\Gamma)$, it follows that $x \to \|u(x, \cdot)\|_{H^1_\delta(\Omega)}$ does not vanish at $+\infty$. Hence we can rule out that $u$ is a solitary wave in any classical sense.

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2.6 Conclusion

Under suitable assumptions on the non-linearity \( f \) and the boundary \( \Gamma \) of \( \Omega \), we have proved in Theorem 2.13 the existence of a minimizer of the variational problem (2.56). Hence we have found a solution of the associated Euler-Lagrange equation and thus a weak solution \( u \) of

\[
\begin{aligned}
\Delta u - \partial_x u &= 0 \quad \text{in } \mathbb{R} \times \Omega, \\
\frac{\partial u}{\partial n} &= \lambda f(u) \quad \text{on } \mathbb{R} \times \Gamma.
\end{aligned}
\]

with \( \lambda > 0 \).

By Theorem 3.8, this solution belongs to \( H^2_\partial((-N,N) \times \Omega) \) for any \( N \in \mathbb{N} \). It follows that the normal derivative of \( u \) on \( \Gamma \) exists at least in the trace sense. Furthermore, standard regularity theory implies \( u \in C^\infty(\mathbb{R} \times \Omega) \). Thus \( u \) is in fact a solution in the classical sense.

The following assumptions were made on the non-linearity \( f \):

1. \( f \in C^1(\mathbb{R}) \) with \( f' \) bounded.
2. \( f(0) = 0 \) and \( 0 \leq f(s)s \) for all \( s \in \mathbb{R} \).
3. \( \exists \Theta > 0 : \Theta F(s) \leq f(s)s \) for all \( s \in \mathbb{R} \).
4. \( \exists \alpha < 1, A > 0 : |f(t)| \leq A|t|^\alpha \) for all \( t \in \mathbb{R} \).
5. \( \exists \delta > 0, \beta > 1, B > 0 : |f(t)| \leq B|t|^\beta \) for all \( |t| \leq \delta \).

Condition 1, 4, and 5 are growth and regularity assumptions. Condition 4 can be reduced to a growth condition at infinity, whereas 5 is a growth condition in 0. Provided a function \( f \) has a sufficiently regular asymptotic behavior at infinity, i.e. does not oscillate wildly for example, condition 3 is typically satisfied provided \( f \) does not converge to 0 at some

![Figure 8: Non-linearity](image)
point. Importantly, none of the conditions prohibits \( f \) from having the characteristics of the boiling curve described in Section 1.1 (see Figure 8).

The equation we originally wanted to solve was (1.3) and not (2.90). This can be achieved by appropriate scaling though. Putting

\[
\Omega^* := \lambda \Omega = \{ \lambda y \mid y \in \Omega \}
\]

\[
\bar{u}(x, y) := u \left( \frac{1}{\lambda} x, \frac{1}{\lambda} y \right), \quad (x, y) \in \mathbb{R} \times \Omega^*
\]

we obtain a solution of

\[
\Delta u - \frac{1}{\lambda} \partial_x u = 0 \quad \text{in } \mathbb{R} \times \Omega^*,
\]

\[
\frac{\partial u}{\partial n} = f(u) \quad \text{on } \mathbb{R} \times \Gamma^*.
\]

Hence we do end up with a solution of (1.3), not in the original domain \( \mathbb{R} \times \Omega \), but in the scaled domain \( \mathbb{R} \times \Omega^* \).

A natural question at this point is to ask what happens if we consider the family of scaled domains \( r \Omega, \ r > 0 \) as starting points? In which domains do we then subsequently obtain solutions? Theorem 3.8 delivers a solution \( u_r \) and an associated Lagrange multiplier \( \lambda_r \) of the corresponding variational problem with \( \mathbb{R} \times r \Omega \) as underlying domain. As in (2.91), scaling \( u_r \) then yields a solution of (1.3) in \( \mathbb{R} \times \lambda_r r \Omega \). In order to characterize this class of domains, we thus need to determine how \( \lambda_r \) depends on \( r \). This, however, is an open question.
3 Regularity

In the previous section, regularity results for solutions of the elliptic PDE
\[
\begin{cases}
\Delta u - \partial_x u = 0 & \text{in } \mathbb{R} \times \Omega,
\end{cases}
\]
(3.1) \[
\frac{\partial u}{\partial n} = g & \text{on } \mathbb{R} \times \Gamma.
\]
were applied. More precisely, we were dealing with solutions \( u \in H^1_2(\mathbb{R} \times \Omega, e^{-x}) \) of the weak formulation

\[
\int_{\mathbb{R} \times \Omega} Du \cdot Dv e^{-x} \, d(x,y) = \int_{\mathbb{R} \times \Gamma} g v e^{-x} \, dS(y) \, dx \quad \forall v \in H^1_2(\mathbb{R} \times \Omega, e^{-x})
\]
(3.2)

of (3.1). Since regularity results for such weak formulations of non-homogeneous Neumann problems are not included in the classical regularity theory for elliptic equations, we shall prove them here. More specifically, regularity results for weak solutions of elliptic equations on divergence form with generalized non-homogeneous Neumann boundary values are established. Moreover, all results are formulated in a framework of unbounded domains.

First, let us briefly state the classical regularity theory, also known as Weyl’s Lemma, for elliptic operators on divergence form. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with a smooth boundary. Consider a differential operator \( A(x, D) \) on \( \Omega \) of the form

\[
A(x, D)\varphi := \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha \beta}(x) D^\beta \varphi) \quad \forall \varphi \in H^m_2(\Omega)
\]
(3.3)

with smooth coefficients \( a_{\alpha \beta} \in C^\infty(\overline{\Omega}) \). Here, and in the rest of the chapter, \( \alpha \) and \( \beta \) shall denote multi-indices of order \( |\alpha| \) and \( |\beta| \), respectively. Assume that the associated bilinear form

\[
a(\varphi, \psi) := \int_\Omega \sum_{|\alpha|,|\beta| \leq m} a_{\alpha \beta}(x) D^\beta \varphi D^\alpha \psi \, dx \quad \forall \varphi, \psi \in H^m_2(\Omega)
\]

is \( H^m_2(\Omega) \)-coercive. This condition is slightly stronger than ellipticity. However, if an elliptic operator \( A \) satisfies Agmon’s condition, by Agmon’s Theorem it is also \( H^m_2(\Omega) \)-coercive. Now consider a functional \( F \in (H^m_2(\Omega))^* \) and a solution \( u \in H^m_2(\Omega) \) of the weak equation

\[
a(u, \psi) = \langle F, \psi \rangle \quad \forall \psi \in H^m_2(\Omega).
\]
The classical Weyl’s Lemma states that if \( F \in (H^m_2(\Omega))^* \) then \( u \in H^{m+k}_2(\Omega) \) (see for example Theorem 20.4 in [Wlo87] or Section 9.7, Chapter 2 in [LM72]).

Now let us consider a problem of the same type as problem (3.2). Consider for this purpose a functional of the type

\[
\langle B, \psi \rangle := \int_{\partial \Omega} \sum_{|\alpha| \leq m-1} b_\alpha(y) D^\alpha \psi \, dS(y) \quad \forall \psi \in H^m_2(\Omega)
\]
(3.4)
with smooth coefficients \( b_\alpha \in C^\infty(\partial\Omega) \). When occurring in a boundary integral as above \( D^\alpha \psi \) is to be understood as the trace of \( D^\alpha \psi \in H^{2-|\alpha|}_2(\Omega) \) in \( L^2(\partial\Omega) \). Problem (3.2) is essentially an equation on the form \[ (3.5) \quad a(u, \psi) = \langle B, \psi \rangle \quad \forall \psi \in H^{m}_2(\Omega). \]

One can consider such problems generalizations of weak non-homogeneous Neumann problems. Let us examine what can be said about the regularity of a solution \( u \) of (3.5) using the standard regularity theory. In order to do so, the regularity of \( B \) must be determined. \( B \) is obviously in \( H^{2-k-\frac{1}{2}}_2(\Omega) \). In fact, due to the trace-operator acting continuously from \( H^{k-\frac{1}{2}}_2(\Omega) \) into \( H^{k-1}(\partial\Omega) \) for \( k > 1 \) and from \( H^{1+\varepsilon}_2(\Omega) \) into \( L^2(\partial\Omega) \), \( B \) belongs to \( (H^{m-\frac{1}{2}+\varepsilon}_2(\Omega))^* \) for all \( \varepsilon > 0 \). Since the trace-operator is not continuous from \( H^1_2(\Omega) \) into \( L^2(\partial\Omega) \), \( B \) will in general not be anymore regular than that. Since \( B \) would have to be at least in \( (H^{m-1}_2(\Omega))^* \) in order to apply the classical Weyl’s Lemma, the standard theory does not deliver any regularity results in this case.

In this section, it is shown how to extend the classical theory to include problems of type (3.5). First, the Weyl’s Lemma is extended to include a larger class of functionals. The proof hereof follows the proof of the classical Weyl’s Lemma found in [Wlo87] with improvements on some of the estimations. Using an appropriate representation of the functional \( B \), regularity for solutions of the generalized weak non-homogeneous Neumann problem (3.5) is then proved as a special case.

In the following, all coefficients and functions are assumed real. The complex case can be dealt with in the exact same manner. Note, however, that in the complex case one would have to impose strong ellipticity on \( A \) in order to apply Agmon’s theorem.

### 3.1 An extension of Weyl’s Lemma

In the first step, the extension of Weyl’s Lemma is proved for a cubic domain. Consider an index \( m \in \mathbb{N} \) and put
\[
W := \{ x \in \mathbb{R}^n \mid |x_i| < 1 \text{ for } i = 1, \ldots, n \},
\]
\[
W^+ := W \cap \{ x \in \mathbb{R}^n \mid x_n > 0 \},
\]
\[
W_\varepsilon := \{ x \in \mathbb{R}^n \mid |x_i| < 1 - \varepsilon \text{ for } i = 1, \ldots, n \} \quad \text{for } 0 < \varepsilon < 1
\]
and
\[
V := \{ u \in H^m_2(W^+) \mid \text{supp } u \subset W \}^{H^m_2},
\]
\[
V_\varepsilon := \{ u \in H^m_2(W^+) \mid \text{supp } u \subset W_\varepsilon \}^{H^m_2}.
\]

\( V \) and \( V_\varepsilon \) are to be considered as subspaces of \( H^m_2(W^+) \) equipped with the norm \( \| \cdot \|_V := \| \cdot \|_{H^m_2} \).
Let $a(\cdot, \cdot)$ be a bilinear form on $V$ of the type

$$
a(u, v) := \int_{W^+} \sum_{|\alpha|, |\beta| \leq m} a_{\alpha\beta}(x) \, D^\alpha u \, D^\beta v \, dx \quad \forall u, v \in V.
$$

Assume that $a(\cdot, \cdot)$ is $V$-coercive, that is,

$$
k_1 \|u\|_V^2 \leq a(u, u) + k_2 \|u\|_{L^2(W^+)}^2 \quad \forall u \in V
$$

for a pair of positive constants $k_1, k_2$. Consider $F \in V^*$ of the type

$$
\langle F, \psi \rangle := \int_{W^+} \sum_{|\alpha| \leq m} g_\alpha(x) \, D^\alpha \psi(x) \, dx \quad \forall \psi \in V
$$

with $g_\alpha \in L^2(W^+)$. The regularity of $F$ of course depends on the regularity of the functions $g_\alpha$. It is now shown that Weyl’s Lemma holds for functionals $F$ with regularity conditions imposed only on highest order $g_\alpha$’s, more specifically with $g_\alpha \in H^1_2(W^+)$ for $|\alpha| = m$. Since such functionals are not necessarily in $(H^{m-1}_2(W^+))^*$, this is indeed an extension of the classical Weyl’s Lemma.

**Lemma 3.1.** Let $F \in V^*$ be a functional of the type (3.8) with

$$
g_\alpha \in L^2(W^+) \text{ for } |\alpha| < m \quad \text{and} \quad g_\alpha \in H^1_2(W^+) \text{ for } |\alpha| = m.
$$

Let $a(\cdot, \cdot)$ be a $V$-coercive bilinear form of type (3.6) with coefficients satisfying

$$
a_{\alpha\beta} \in C^1(W^+) \quad \forall |\alpha|, |\beta| \leq m.
$$

If $u \in V$ satisfies

$$
a(u, \psi) = \langle F, \psi \rangle \quad \forall \psi \in V_\varepsilon
$$

then $\chi u \in H^{m+1}_2(W^+)$ for all $\chi \in C^\infty_\varepsilon(W_{3\varepsilon})$.

**Proof.**

Let $\Delta_h^i$ denote the difference quotient

$$
\Delta_h^i u(x) := \frac{u(x + he_i) - u(x)}{h}
$$

of size $h$ in the $x_i$-direction. Fix $1 \leq i \leq n - 1$. Then $e_i$ is a tangential direction with respect to the boundary $\{x \in W^+ \mid x_n = 0\}$. For $\varphi \in V_{3\varepsilon}$ and $|h| < \varepsilon$ it holds that

$$
a(\Delta_h^i \chi u, \varphi) = \int_{W^+} \sum_{|\alpha|, |\beta| \leq m} -\Delta_h^i a_{\alpha\beta}(x) \, D^\alpha [\chi u](x) \, D^\beta \varphi(x - he_i) \, dx
$$

$$
+ \int_{W^+} \sum_{|\alpha|, |\beta| \leq m} a_{\alpha\beta}(x) \, D^\alpha [\chi u](x) \, D^\beta (-\Delta_h^i) \varphi(x) \, dx.
$$
The first integral on the right-hand side above can be estimated by

\[
| \int_{W^+} \sum_{|\alpha|,|\beta| \leq m} -\Delta_h^i a_{\alpha\beta}(x) D^\alpha [\chi u](x) D^\beta \varphi(x - he_i) \, dx |
\]

(3.14)

\[
\leq \sum_{|\alpha|,|\beta| \leq m} \|\Delta_h^i a_{\alpha\beta}(x)\|_\infty \|D^\alpha [\chi u]\|_{L^2(W^+)} \|D^\beta \varphi\|_{L^2(W^+)}
\]

\[
\leq C \sum_{|\alpha|,|\beta| \leq m} \|a_{\alpha\beta}(x)\|_{C^1(\overline{W^+})} \|u\|_V \|\varphi\|_V \leq C \|u\|_V \|\varphi\|_V .
\]

Applying the chain rule consecutively, one obtains for the second integral on the right-hand side in (3.13) the identity

\[
\int_{W^+} \sum_{|\alpha|,|\beta| \leq m} a_{\alpha\beta}(x) D^\alpha [\chi u](x) D^\beta (-\Delta_h^i) \varphi(x) \, dx
\]

(3.15)

\[
= - \int_{W^+} \sum_{|\alpha|,|\beta| \leq m} a_{\alpha\beta}(x) D^\alpha u(x) D^\beta [\chi \Delta_h^i \varphi](x) \, dx
\]

\[
+ \int_{W^+} \sum_{|\alpha| \leq m} b_{\alpha\beta}(x) D^\alpha u(x) D^\beta [\Delta_h^i \varphi](x) \, dx
\]

\[
+ \int_{W^+} \sum_{|\alpha| \leq m-1} c_{\alpha\beta}(x) D^\alpha u(x) D^\beta [\Delta_h^i \varphi](x) \, dx
\]

with \(b_{\alpha\beta}\) and \(c_{\alpha\beta}\) being functions of the type \(D^\gamma \chi a_{\alpha\beta}\). It follows that \(b_{\alpha\beta}, c_{\alpha\beta} \in C^1(\overline{W^+})\). For \(|\beta| \leq m - 1\), one has \(D^\beta \varphi \in H^2(W^+)\). Since \(1 \leq i \leq n - 1\) and \(\varphi \in V_{2e}\), the difference quotient \(\Delta_h^i D^\beta \varphi\) is in \(L^2(W^+)\) for \(|h| < \varepsilon\) and \(\|\Delta_h^i D^\beta \varphi\|_{L^2(W^+)} \leq C \|D^\beta \varphi\|_{H^2(W^+)}\) for all \(|h| < \varepsilon\). Hence the second integral in (3.15) can be estimated by

\[
| \int_{W^+} \sum_{|\alpha| \leq m} b_{\alpha\beta} D^\alpha u D^\beta [\Delta_h^i \varphi] \, dx |
\]

(3.16)

\[
\leq \sum_{|\alpha| \leq m} \|b_{\alpha\beta}\|_\infty \|D^\alpha u\|_{L^2(W^+)} \|\Delta_h^i D^\beta \varphi\|_{L^2(W^+)}
\]

\[
\leq \sum_{|\alpha|,|\beta| \leq m} \|b_{\alpha\beta}\|_{C^1(\overline{W^+})} \|D^\alpha u\|_{L^2(W^+)} C \|D^\beta \varphi\|_{L^2(W^+)} \leq C \|u\|_V \|\varphi\|_V .
\]

Similarly, since \(c_{\alpha\beta} D^\alpha u \in H^2(W^+)\) for \(|\alpha| \leq m - 1\), one has for the last integral in (3.15)
the estimate

\[ \left| \int_{W^+} \sum_{|\alpha| \leq m-1 \atop |\beta| \leq m} c_{\alpha \beta} D^\alpha u \ D^\beta [\Delta^i_{-h} \varphi] \ dx \right| = \left| \int_{W^+} \sum_{|\alpha| \leq m-1 \atop |\beta| \leq m} \Delta^i_h [c_{\alpha \beta} D^\alpha u] \ D^\beta \varphi \ dx \right| \]

\[ \leq \sum_{|\alpha| \leq m-1 \atop |\beta| \leq m} \| \Delta^i_h [c_{\alpha \beta} D^\alpha u] \|_{L^2(W^+)} \| D^\beta \varphi \|_{L^2(W^+)} \]

\[ \leq \sum_{|\alpha| \leq m-1 \atop |\beta| \leq m} C \| D_i [c_{\alpha \beta} D^\alpha u] \|_{L^2(W^+)} \| D^\beta \varphi \|_{L^2(W^+)} \]

\[ \leq \sum_{|\alpha|, |\beta| \leq m} C \| c_{\alpha \beta} \|_{C^1(\overline{W^+})} \| D^\alpha u \|_{L^2(W^+)} \| D^\beta \varphi \|_{L^2(W^+)} \leq C \| u \|_V \| \varphi \|_V . \]

(3.17)

Now recognizing the first integral on the right-hand side in (3.15) as \(-a(u, \chi \Delta^i_{-h} \varphi)\), by (3.13), (3.14), (3.15), (3.16), and (3.17) one has

\[ a(\Delta^i_h [\chi u], \varphi) = -a(u, \chi \Delta^i_{-h} \varphi) + I(u, \varphi, h) , \]

whereby

\[ |I(u, \varphi, h)| \leq C \| u \|_V \| \varphi \|_V . \]

(3.19)

Since \( u \) is a solution of (3.11) and \( \chi \Delta^i_{-h} \varphi \in V_\varepsilon \), it follows that

\[ a(u, \chi \Delta^i_{-h} \varphi) = \langle F, \chi \Delta^i_{-h} \varphi \rangle = \int_{W^+} \sum_{|\alpha| \leq m} g_\alpha \ D^\alpha [\chi \Delta^i_{-h} \varphi] \ dx \]

\[ = \int_{W^+} \sum_{|\alpha| \leq m} \tilde{g}_\alpha \Delta^i_{-h} D^\alpha \varphi \ dx \]

(3.20)

with \( \tilde{g}_\alpha \) being functions of the type \( \sum_{\lambda, \gamma} k_{\lambda, \gamma} g_\lambda D^\gamma \chi \) (\( k_{\lambda, \gamma} \) denoting constants) for \( |\alpha| < m \) and \( \tilde{g}_\alpha = \chi g_\alpha \) for \( |\alpha| = m \). Consequently \( \tilde{g}_\alpha \in L^2(W^+) \) for \( |\alpha| < m \) and \( \tilde{g}_\alpha \in H^1_0(W^+) \) for
\(|\alpha| = m\). Thus

\[
|a(u, \chi \Delta^i \varphi)| = \left| \int_{W^+} \sum_{|\alpha| < m} \tilde{g}_\alpha \Delta^i_{-h} D^\alpha \varphi \, dx - \int_{W^+} \sum_{|\alpha| = m} \Delta^i_{h} \tilde{g}_\alpha \, D^\alpha \varphi \, dx \right|
\]

\[
\leq \sum_{|\alpha| < m} \|\tilde{g}_\alpha\|_{L^2(W^+)} \|\Delta^i_{-h} D^\alpha \varphi\|_{L^2(W^+)} + \sum_{|\alpha| = m} \|\Delta^i_{h} \tilde{g}_\alpha\|_{L^2(W^+)} \|D^\alpha \varphi\|_{L^2(W^+)}
\]

(3.21)

\[
\leq \sum_{|\alpha| \leq m} C \|D^\alpha \varphi\|_{L^2(W^+)} + \sum_{|\alpha| = m} C' \|\tilde{g}_\alpha\|_{H^2(W^+)} \|D^\alpha \varphi\|_{L^2(W^+)}
\]

\[
\leq C \|\varphi\|_V .
\]

Now since \(\Delta^i_h [\chi u] \in V_2\) for \(|h| < \varepsilon\), one can choose \(\varphi = \Delta^i_h [\chi u]\) in (3.18). Using (3.19), (3.21), and the coerciveness of \(a\), it then follows that

(3.22)

\[
k_1 \|\Delta^i_h [\chi u]\|_V^2 - k_2 \|\Delta^i_h [\chi u]\|_{L^2(W^+)}^2 \leq C_1 \|\Delta^i_h [\chi u]\|_V + C_2 \|u\|_V \|\Delta^i_h [\chi u]\|_V .
\]

Estimating

\[
\|\Delta^i_h [\chi u]\|_{L^2(W^+)}^2 \leq C \|\chi u\|_{H^2(W^+)}^2 \leq C_3 \|u\|_V^2
\]

in (3.22) yields

(3.23)

\[
k_1 \|\Delta^i_h [\chi u]\|_V^2 - (C_1 + C_2 \|u\|_V) \|\Delta^i_h [\chi u]\|_V - k_2 C_3 \|u\|_V^2 \leq 0 .
\]

Consider the polynomial

\[
p(\lambda) := k_1 \lambda^2 - (C_1 + C_2 \|u\|_V) \lambda - k_2 C_3 \|u\|_V .
\]

For a zero \(\lambda_0\) of \(p\) one has the estimate \(\lambda_0 \leq C_4 (1 + \|u\|_V)\). Hence from (3.23) one can deduce

\[
\|\Delta^i_h [\chi u]\|_V \leq C_4 (1 + \|u\|_V) .
\]

In particular

\[
\|\Delta^i_h D^\alpha [\chi u]\|_{L^2(W^+)} \leq C_4 (1 + \|u\|_V)
\]

for all \(|\alpha| = m\). Since this estimate is valid for all \(|h| \leq \varepsilon\), the existence of the weak i’th derivative of \(D^\alpha [\chi u]\) as an element of \(L^2(W^+)\) with

(3.24)

\[
\|D_i D^\alpha [\chi u]\|_{L^2(W^+)} \leq C_4 (1 + \|u\|_V)
\]

follows.

Now the weak differentiability of \(D^\alpha [\chi u]\) in the non-tangential direction \(e_n\) must be established for all \(\alpha\) with \(|\alpha| = m\). If \(\alpha \neq (0, \ldots, m)\) one can find an \(1 \leq i < n\) such that \(D_n[D^\alpha [\chi u]] = D_i[D^\tilde{\alpha} [\chi u]]\) with \(|\tilde{\alpha}| = |\alpha| = m\). Hence in this case the existence of
The $V$-coercion of $a(\cdot, \cdot)$ implies $H^m_2(W^+)$-coercion. Hence by Gårding's Theorem, the differential operator $A$ determined by the coefficients $\{a_{\alpha\beta}\}_{|\alpha|+|\beta| \leq m}$ is elliptic. In particular $a_{(0,\ldots,m)(0,\ldots,m)}(x) > 0$ for all $x \in W^+$. Since by assumption the coefficient $a_{(0,\ldots,m)(0,\ldots,m)}$ is sufficiently regular, the existence of the weak derivative $D_n[a_{(0,\ldots,m)(0,\ldots,m)}D_{(0,\ldots,m)}[\chi u]]$ as an element of $L^2(W^+)$ thus implies $D_n[a_{(0,\ldots,m)(0,\ldots,m)}D_{(0,\ldots,m)}[\chi u]] \in L^2(W^+)$. 

In order to show $D_n[a_{(0,\ldots,m)(0,\ldots,m)}D_{(0,\ldots,m)}[\chi u]] \in L^2(W^+)$ we will apply Proposition 3.2. For this purpose, define on $C^\infty_c(W^+)$ the functional

$$\langle H, \varphi \rangle := \int_{W^+} a_{(0,\ldots,m)(0,\ldots,m)}(x) D_{(0,\ldots,m)}[\chi u] D_{(0,\ldots,m+1)} \varphi \, dx \quad \forall \varphi \in C^\infty_c(W^+) .$$

One has

$$\langle H, \varphi \rangle = \int_{W^+} a_{(0,\ldots,m)(0,\ldots,m)}(x) \chi D_{(0,\ldots,m)} u D_{(0,\ldots,m+1)} \varphi \, dx$$

$$+ \int_{W^+} a_{(0,\ldots,m)(0,\ldots,m)} \sum_{j=1}^m \binom{m}{j} D_{(0,\ldots,m-j)} \chi D_{(0,\ldots,m-j)} u D_{(0,\ldots,m+1)} \varphi \, dx .$$

(3.25)

By partial integration, the last integral above can be estimated by

$$| \int_{W^+} a_{(0,\ldots,m)(0,\ldots,m)} \sum_{j=1}^m \binom{m}{j} D_{(0,\ldots,m-j)} \chi D_{(0,\ldots,m-j)} u D_{(0,\ldots,m+1)} \varphi \, dx |$$

$$= | \sum_{j=1}^m \binom{m}{j} \int_{W^+} D_n[a_{(0,\ldots,m)(0,\ldots,m)} D_{(0,\ldots,m-j)} \chi D_{(0,\ldots,m-j)} u] D_{(0,\ldots,m)} \varphi \, dx |$$

$$\leq C \|u\|_V \|\varphi\|_{H^m_2(W^+)} .$$

The first integral on the right-hand side in (3.25) can be written as

$$\int_{W^+} a_{(0,\ldots,m)(0,\ldots,m)}(x) \chi D_{(0,\ldots,m)} u D_{(0,\ldots,m+1)} \varphi \, dx$$

$$= \int_{W^+} a_{(0,\ldots,m)(0,\ldots,m)} D_{(0,\ldots,m)} u D_{(0,\ldots,m)}[\chi D_n \varphi] \, dx$$

$$- \int_{W^+} a_{(0,\ldots,m)(0,\ldots,m)} D_{(0,\ldots,m)} u \sum_{j=1}^m \binom{m}{j} D_{(0,\ldots,m-j)} \chi D_{(0,\ldots,m+1-j)} \varphi \, dx .$$

(3.27)

In the last integral above only $\varphi$-derivatives of order less than or equal to $m$ occur. Hence

$$| \int_{W^+} a_{(0,\ldots,m)(0,\ldots,m)} D_{(0,\ldots,m)} u \sum_{j=1}^m \binom{m}{j} D_{(0,\ldots,m-j)} \chi D_{(0,\ldots,m+1-j)} \varphi \, dx |$$

$$\leq C \|u\|_V \|\varphi\|_{H^m_2(W^+)} .$$

(3.28)
Recalling the definition of \( a(\cdot, \cdot) \), the first integral on the right hand side in (3.27) evaluates to

\[
\int_{W^+} a_{(0,\ldots,m)(0,\ldots,m)} D^{(0,\ldots,m)} u \, D^{(0,\ldots,m)}[\chi D_n \varphi] \, dx
\]

\[
= a(u, \chi D_n \varphi) - \int_{W^+} \sum_{\|\alpha\|,\|\beta\| \leq m, \alpha, \beta \neq (0,\ldots,m)} a_{\alpha\beta} D^\alpha u \, D^{\beta}[\chi D_n \varphi] \, dx .
\]

Now applying the chain rule on each element of the sum above one obtains

\[
\int_{W^+} a_{(0,\ldots,m)(0,\ldots,m)} D^{(0,\ldots,m)} u \, D^{(0,\ldots,m)}[\chi D_n \varphi] \, dx
\]

(3.29)

\[
= a(u, \chi D_n \varphi) - \int_{W^+} \sum_{\|\alpha\|,\|\beta\| \leq m, \alpha, \beta \neq (0,\ldots,m)} a_{\alpha\beta} D^\alpha [\chi u] \, D^{\beta}[D_n \varphi] \, dx + I(u, \varphi)
\]

with

\[
|I(u, \varphi)| \leq C \|u\|_V \|\varphi\|_{H^2(W^+)} .
\]

Consider an element \( a_{\alpha\beta} D^\alpha [\chi u] D^{\beta}[D_n \varphi] \) of the sum in (3.29). If \(|\beta| < m\), one can directly estimate

\[
| \int_{W^+} a_{\alpha\beta} D^\alpha [\chi u] \, D^{\beta}[D_n \varphi] \, dx | \leq C \|u\|_V \|\varphi\|_{H^2(W^+)} .
\]

Now consider an index \( \beta \) of the sum with \(|\beta| = m\). Since \( \beta \neq (0,\ldots,m) \), \( D^\beta \) contains at least one derivative in a tangential direction \( e_i \) for some \( 1 \leq i < n \). Having already established the existence of the weak derivative \( D_i[D^\alpha[\chi u]] \) as an element of \( L^2(W^+) \) for \(|\alpha| \leq m\), it follows by partial integration that

\[
\int_{W^+} a_{\alpha\beta} D^\alpha [\chi u] \, D^{\beta}[D_n \varphi] \, dx = - \int_{W^+} D_i[a_{\alpha\beta} D^\alpha[\chi u]] \, D^{\beta}[D_n \varphi] \, dx
\]

with \(|\beta| = m - 1\). Thus

\[
| \int_{W^+} a_{\alpha\beta} D^\alpha [\chi u] \, D^{\beta}[D_n \varphi] \, dx | \leq \int_{W^+} | D_i[a_{\alpha\beta} D^\alpha[\chi u]] \, D^{\beta}[D_n \varphi] | \, dx + \int_{W^+} | a_{\alpha\beta} D_i[D^\alpha[\chi u]] \, D^{\beta}[D_n \varphi] | \, dx \leq \|a_{\alpha\beta}\|_{C^1(W^+)} \left( \|D^\alpha[\chi u]\|_{L^2(W^+)} + \|D_i[D^\alpha[\chi u]]\|_{L^2(W^+)} \right) C \|\varphi\|_{H^m(W^+)} .
\]
Inserting (3.24) now yields
\[
| \int_{W^+} a_{\alpha \beta} D^\alpha [\chi u] D^\beta [D_n \varphi] \, dx | \leq C \left( 1 + \| u \|_V \right) \| \varphi \|_{H^m_2(W^+)}.
\]

Hence for the sum in (3.29) one has
\[
(3.30) \quad | \int_{W^+} \sum_{|\alpha|, |\beta| \leq m, \alpha, \beta \neq (0, \ldots, m)} a_{\alpha \beta} D^\alpha u D^\beta [\chi D_n \varphi] \, dx | \leq C \left( 1 + \| u \|_V \right) \| \varphi \|_{H^m_2(W^+)}.
\]

Since \( u \) is a solution of (3.11), one has for the first term on the right-hand side in (3.29) the identity
\[
(3.31) \quad a(u, \chi D_n \varphi) = \langle F, \chi D_n \varphi \rangle = \int_{W^+} \sum_{|\alpha| \leq m} g_\alpha(x) D^\alpha [\chi D_n \varphi] \, dx.
\]

For \( |\alpha| < m \), one can directly estimate
\[
(3.32) \quad | \int_{W^+} g_\alpha(x) D^\alpha [\chi D_n \varphi] \, dx | \leq C \| g_\alpha \|_{L^2(W^+)} \| \varphi \|_{H^m_2(W^+)}.
\]

For \( |\alpha| = m \), \( g_\alpha \in H^1_2(W^+) \) by assumption. Hence choosing \( 1 \leq i \leq n \) such that the \( i \)’th component of \( \alpha \) is not zero, a partial integration yields
\[
(3.33) \quad | \int_{W^+} g_\alpha(x) D^\alpha [\chi D_n \varphi] \, dx | = | \int_{W^+} D_i g_\alpha(x) D^{\alpha_i} [\chi D_n \varphi] \, dx | \quad ; \quad |\alpha| = m - 1
\]
\[
\leq C \| g_\alpha \|_{H^1_2(W^+)} \| \varphi \|_{H^m_2(W^+)}.
\]

It follows that
\[
(3.34) \quad |a(u, \chi D_n \varphi)| \leq C \left( \sum_{|\alpha| < m} \| g_\alpha \|_{L^2(W^+)} + \sum_{|\alpha| = m} \| g_\alpha \|_{H^1_2(W^+)} \right) \| \varphi \|_{H^m_2(W^+)}.
\]

Finally, by (3.25),(3.26),(3.27),(3.28),(3.29), (3.30), and (3.34) one has
\[
(3.35) \quad | \langle H, \varphi \rangle | \leq C \| \varphi \|_{H^m_2(W^+)} \quad \forall \varphi \in C_c^\infty(W^+).
\]

Consequently, Proposition 3.2 implies the existence of the \( n \)’th weak derivative of \( a_{(0, \ldots, m)(0, \ldots, m)} D^{(0, \ldots, m)} [\chi u] \) as an element of \( L^2(W^+) \). This completes the proof. \( \square \)

In the proof above, the following proposition was used. As formulated below, the proposition and an elegant proof hereof can be found in [Wlo87].
3.1 An extension of Weyl’s Lemma

Proposition 3.2. Let \( m \in \mathbb{N} \) and \( f \in L^2(W^+) \) with weak derivatives \( D_i f \in L^2(W^+) \) for \( i = 1, \ldots, n-1 \). Consider on \( C_c^\infty(W^+) \) the functional

\[
(H, \varphi) := \int_{W^+} f \, D_n^m \varphi \, dx \quad \forall \varphi \in C_c^\infty(W^+)
\]

If

\[
|h(H, \varphi)| \leq C \| \varphi \|_{H_2^{m-1}(W^+)} \quad \forall \varphi \in C_c^\infty(W^+)
\]

then the weak derivative \( D_n f \) exists as an element of \( L^2(W^+) \).

Proof.
See Proposition 20.3 in [Wlo87]. \( \square \)

By iterating the arguments in the proof of Lemma 3.1, we can now obtain the following higher-order regularity result.

Lemma 3.3. Let \( k \in \mathbb{N} \) and \( F \in V^* \) be a functional of the type (3.8) with

\[
g_\alpha \in H_2^{\max(0,k+|\alpha|-m)}(W^+) \quad \forall |\alpha| \leq m
\]

Let \( a(\cdot, \cdot) \) be a \( V \)-coercive bilinear form of type (3.6) with coefficients

\[
a_{\alpha\beta} \in C^k(W^+) \quad \forall |\alpha|, |\beta| \leq m
\]

If \( u \in V \) satisfies

\[
a(u, \psi) = \langle F, \psi \rangle \quad \forall \psi \in V_c
\]

then \( \chi u \in H_2^{m+k}(W^+) \) for all \( \chi \in C_c^\infty(W_{3\varepsilon}) \).

Proof.
As indicated above, the lemma is proved by induction in \( k \). Lemma 3.1 provides the result for \( k = 1 \). Assume now that the lemma holds for some \( k \geq 1 \). Consider \( F \in V^* \) of type (3.8) with \( g_\alpha \in H_2^{\max(0,k+1+|\alpha|-m)}(W^+) \). Furthermore assume \( a_{\alpha\beta} \in C^{k+1}(W^+) \) for all \( |\alpha|, |\beta| \leq m \). Let \( u \in V \) be an element satisfying (3.40) and \( \chi \in C_c^\infty(W_{3\varepsilon}) \). Now \( \chi u \in H_2^{m+k+1}(W^+) \) must be shown.

Consider first a multi-index \( \gamma \) with \( |\gamma| = k \) containing non-zero components only in a tangential direction (with respect to the boundary \( \{ x \in W^+ \mid x_n = 0 \} \)), that is, \( \gamma = (\gamma_1, \ldots, \gamma_n-1, 0) \). Since by assumption the lemma holds for \( k \), \( \chi u \in H_2^{m+k}(W^+) \) and hence \( D^\gamma[\chi u] \in H_2^m(W^+) \) follows. Let \( \varphi \in V_{2\varepsilon} \cap H_2^{m+k}(W^+) \), \( |h| < \varepsilon \), and \( 0 \leq i \leq n-1 \). By partial integration and estimations equivalent to those made in the proof of Lemma 3.1, one can prove

\[
a(\Delta_i^i D^\gamma[\chi u], \varphi) = (-1)^{|\gamma|+1} a(u, \chi \Delta_i^i h D^\gamma \varphi) + I(u, \varphi, h)
\]

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with

\[ |I(u, \varphi, h)| \leq C \|u\|_{m+k} \|\varphi\|_m. \]  

(3.42)

Here \(\|\cdot\|_I\) denotes the \(H^1_2(W^+)\)-norm. Note that \(a_{\alpha\beta} \in C^k(\overline{W^+})\) is essential in order to obtain (3.42). Also note that since \(\varphi \in V_{2\varepsilon}\) and therefore \(\Delta^+_{-h} \varphi \in \mathcal{V}_{\varepsilon}\), no boundary terms occur when integrating partially in tangential directions. Since \(u\) is a solution of (3.40) and \(\chi \Delta^+_h D^\gamma \varphi \in V_\varepsilon\), it follows that

\[
a(u, \chi \Delta^+_h D^\gamma \varphi) = \langle F, \chi \Delta^+_h D^\gamma \varphi \rangle
\]

\[
= \int_{W^+} \sum_{|\alpha| \leq m} g_\alpha D^\alpha [\chi \Delta^+_h D^\gamma \varphi] \, dx
\]

(3.43)

\[
= \int_{W^+} \sum_{|\alpha| \leq m} \tilde{g}_\alpha D^\alpha \Delta^+_h D^\gamma \varphi \, dx
\]

with \(\tilde{g}_\alpha \in H^\max(0,k+1+|\alpha|-m)(W^+)\). Consequently, partial integration yields

\[
a(u, \chi \Delta^+_h D^\gamma \varphi) = \int_{W^+} \sum_{|\alpha| < m} (-1)^{|\tilde{\alpha}|} D^{\tilde{\alpha}} \tilde{g}_\alpha \Delta^+_h D^{\gamma+\tilde{\alpha}} \varphi \, dx
\]

(3.44)

\[
+ (-1)^{|\tilde{\alpha}|+1} \int_{W^+} \sum_{|\alpha| = m} \Delta^+_h D^\gamma \tilde{g}_\alpha D^{\alpha} \varphi \, dx
\]

whereby each \(\tilde{\alpha}\) is chosen such that \(\tilde{\alpha} \leq \gamma\) (and hence contains only non-zero components in tangential directions) and \(|\tilde{\alpha}| = \max(0, k + 1 + |\alpha| - m)\). It follows that

\[
|a(u, \chi \Delta^+_h D^\gamma \varphi)| \leq \sum_{|\alpha| < m} C \|\tilde{g}_\alpha\|_{\max(0,k+1+|\alpha|-m)} \|\varphi\|_m + \sum_{|\alpha| = m} C \|\tilde{g}_\alpha\|_{k+1} \|\varphi\|_m \leq C \|\varphi\|_m.
\]

(3.45)

Hence by (3.41)

\[
a(\Delta^+_h D^\gamma [\chi u], \varphi) \leq C_1 \|\varphi\|_m + C_2 \|u\|_{m+k} \|\varphi\|_m
\]

(3.46)

for all \(\varphi \in V_{2\varepsilon} \cap H^m_{2}(W^+)\). Since \(V_{2\varepsilon} \cap H^m_{2}(W^+)\) lies dense in \(V_{2\varepsilon}\), the inequality extends to all functions \(\varphi \in V_{2\varepsilon}\). Thus one can choose \(\varphi = \Delta^+_h D^\gamma [\chi u]\) in (3.46). The coerciveness of \(a(\cdot, \cdot)\) then implies

\[
k_1 \|\Delta^+_h D^\gamma [\chi u]\|_m^2 - k_2 \|\Delta^+_h D^\gamma [\chi u]\|_{L^2(W^+)}^2 \leq (C_1 + C_2 \|u\|_{m+k}) \|\Delta^+_h D^\gamma [\chi u]\|_m.
\]

(3.47)

Estimating

\[
\|\Delta^+_h D^\gamma [\chi u]\|_{L^2(W^+)}^2 \leq C_3 \|u\|_{m+k}^2
\]

(3.48)

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thus yields

\[(3.49)\quad k_1 \| \Delta_h^j \mathcal{D}^\gamma [\chi u] \|_m^2 - (C_1 + C_2 \| u \|_{m+k}) \| \Delta_h^j \mathcal{D}^\gamma [\chi u] \|_m - k_2 C_3 \| u \|_{m+k}^2 \leq 0.\]

By the same argument as in the proof of Lemma 3.1, it follows from (3.49) that

\[(3.50)\quad \| \Delta_h^j \mathcal{D}^\gamma [\chi u] \|_m \leq C_4 (1 + \| u \|_{m+k}).\]

From this we deduce the existence of $D_i \mathcal{D}^{\alpha+\gamma} [\chi u]$ as an element of $L^2(W^+)$ with

\[(3.51)\quad \| D_i \mathcal{D}^{\alpha+\gamma} [\chi u] \|_{L^2(W^+)} \leq C_4 (1 + \| u \|_{m+k})\]

for all $\alpha$ with $|\alpha| = m$. Consequently, we now have $D^\gamma \mathcal{D}^\alpha [\chi u] \in L^2(W^+)$ for $|\alpha| = m$ and $|\gamma| = k + 1$ with $\gamma$ of type $\gamma = (\gamma_1, \ldots, \gamma_{n-1}, 0)$.

Now the same must be shown for $\gamma$ with $|\gamma| = k + 1$ and non-zero $n$'th component. Consider first $\gamma = (\gamma_1, \ldots, \gamma_{n-1}, 1)$. Put $\gamma' = (\gamma_1, \ldots, \gamma_{n-1}, 0)$. For $|\alpha| = m$ with $\alpha \neq (0, \ldots, m)$, $D^\gamma \mathcal{D}^\alpha [\chi u] \in L^2(W^+)$ follows by the argument above. In order to show $D^\gamma \mathcal{D}^{(0, \ldots, m)} [\chi u] \in L^2(W^+)$, define on $C^\infty_c(W^+)$ the functional

\[(3.52)\quad \langle H_1, \varphi \rangle := \int_{W^+} a_{(0, \ldots, m)(0, \ldots, m)} D^\gamma \mathcal{D}^{(0, \ldots, m)} [\chi u] D^{(0, \ldots, m+1)} \varphi \, dx \quad \forall \varphi \in C^\infty_c(W^+).\]

Since by assumption $a_{(0, \ldots, m)(0, \ldots, m)} \in C^{k+1}(\overline{W^+})$, partial integration yields

\[(3.53)\quad \langle H_1, \varphi \rangle = (-1)^{|\gamma|} \int_{W^+} a_{(0, \ldots, m)(0, \ldots, m)} D_{(0, \ldots, m)} [\chi u] D^\gamma \mathcal{D}^{(0, \ldots, m+1)} \varphi \, dx + I_1(u, \varphi)\]

with

\[(3.54)\quad |I_1(u, \varphi)| \leq C \| u \|_{m+k} \| \varphi \|_m.\]

Consecutively using the chain rule it follows that

\[(3.55)\quad \langle H_1, \varphi \rangle = (-1)^{|\gamma|} \int_{W^+} a_{(0, \ldots, m)(0, \ldots, m)} D_{(0, \ldots, m)} u D^\gamma \mathcal{D}^{(0, \ldots, m+1)} [\chi \varphi] \, dx + I_2(u, \varphi)\]

with

\[(3.56)\quad |I_2(u, \varphi)| \leq C \| u \|_{m+k} \| \varphi \|_m.\]

Recalling the definition of $a(\cdot, \cdot)$, the first integral in (3.55) can be written as

\[(3.57)\quad \int_{W^+} a_{(0, \ldots, m)(0, \ldots, m)} D_{(0, \ldots, m)} u D^\gamma \mathcal{D}^{(0, \ldots, m+1)} [\chi \varphi] \, dx = a(u, D^\gamma \mathcal{D}_n [\chi \varphi]) - \int_{W^+} \sum_{\alpha, \beta \leq m \atop \alpha, \beta \neq (0, \ldots, m)} a_{\alpha \beta} D^\alpha u D^\beta D^\gamma \mathcal{D}_n [\chi \varphi] \, dx.\]
As in the proof of Lemma 3.1, one can use (3.51) to estimate the last integral in (3.57) to obtain

\[
| \int_{W^+} \sum_{\alpha, \beta \leq m, \alpha, \beta \neq (0, \ldots, m)} a_{\alpha \beta} D^\alpha u D^\beta D^\gamma D_n [\chi \varphi] \, dx | \leq C (1 + \|u\|_{m+k}) \|\varphi\|_m .
\]  

Since \( u \) is a solution of (3.40), one has for the first term on the right-hand side in (3.57) the identity

\[
a(u, D^\gamma D_n [\chi \varphi]) = \langle F, D^\gamma D_n [\chi \varphi] \rangle
\]

\[
= \int_{W^+} \sum_{|\alpha| < m} g_\alpha D^\alpha D^\gamma D_n [\chi \varphi] \, dx + \int_{W^+} \sum_{|\alpha| = m} g_\alpha D^\alpha D^\gamma D_n [\chi \varphi] \, dx
\]

\[
(3.59)
\]

\[
= \int_{W^+} \sum_{|\alpha| < m} (-1)^{|\alpha|} D^{\tilde{\alpha}} g_\alpha D^{\alpha + \gamma - \tilde{\alpha}} D_n [\chi \varphi] \, dx + \sum_{|\alpha| = m} (-1)^{|\gamma| + 1} \int_{W^+} D_n D^{\gamma} g_\alpha D^\alpha [\chi \varphi] \, dx
\]

whereby each \( \tilde{\alpha} \) is chosen such that \( \tilde{\alpha} \leq \gamma \) and \( |\tilde{\alpha}| = \max(0, k + |\alpha| + 1 - m) \). It follows that

\[
|a(u, D^\gamma D_n [\chi \varphi])| \leq C \left( \sum_{|\alpha| < m} \|g_\alpha\|_{\max(0, k + |\alpha| + 1 - m)} \|\varphi\|_m + \sum_{|\alpha| = m} \|g_\alpha\|_{\|k+1\|} \|\varphi\|_m \right).
\]  

Thus by (3.55), (3.56), (3.57), (3.58) and (3.60)

\[
|\langle H_1, \varphi \rangle| \leq C \|\varphi\|_m \quad \forall \varphi \in C_c^\infty (W^+) .
\]  

Applying Proposition 3.2 we obtain \( D_n \left[ a_{(0, \ldots, m)(0, \ldots, m)} D^\gamma D^{(0, \ldots, m)} [\chi u] \right] \in L^2(W^+) \) from which \( D^\gamma D^{(0, \ldots, m)} [\chi u] \in L^2(W^+) \) follows as in the proof of Lemma 3.1.

Now consider \( \gamma = (\gamma_1, \ldots, \gamma_{n-1}, j) \) with \( j > 1 \) and \( |\gamma| = k + 1 \). Assume that \( D^\gamma D^{(0, \ldots, m)} [\chi u] \in L^2(W^+) \) has been established for all \( \tilde{\gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n-1}, j-1) \) with \( |\tilde{\gamma}| = k + 1 \). It follows directly that \( D^\gamma D^\alpha [\chi u] \in L^2(W^+) \) for \( |\alpha| = m \) with \( \alpha \neq (0, \ldots, m) \). Putting \( \gamma' = (\gamma_1, \ldots, \gamma_{n-1}, j - 1) \),

\[
(3.62) \quad \langle H_j, \varphi \rangle := \int_{W^+} a_{(0, \ldots, m)(0, \ldots, m)} D^\gamma D^{(0, \ldots, m)} [\chi u] D^{(0, \ldots, m+1)} \varphi \, dx \quad \forall \varphi \in C_c^\infty (W^+) ,
\]

and repeating the argument above with \( H_1 \) replaced by \( H_j \) delivers \( D^\gamma D^{(0, \ldots, m)} [\chi u] \in L^2(W^+) \). Hence by induction in \( j \), \( D^\gamma D^\alpha [\chi u] \in L^2(W^+) \) for all \( \gamma \) with \( |\gamma| = k + 1 \) and all \( |\alpha| = m \).
The extension of Weyl’s Lemma can now be proved for a sufficiently smooth domain \( \Omega \subset \mathbb{R}^n \). The proof is based on Lemma 3.3 and a localization argument. We hereby obtain regularity results on each compact subset of \( \Omega \). If \( \Omega \) is bounded, global regularity then follows. If \( \Omega \) is unbounded, however, this will not necessarily be the case. In order to express the regularity properties for general, possibly unbounded, domains, we thus introduce the space

\[
H^{m}_{2,B-loc}(\Omega) := \{ u \in L^2_{loc}(\Omega) \mid \chi u \in H^m_2(\Omega) \text{ for all } \chi \in C_c^\infty(\mathbb{R}^n) \}.
\]

If we equip \( H^{m}_{2,B-loc}(\Omega) \) with the topology induced by the family of seminorms

\[
\| \cdot \|_B := \| \cdot \|_{H^2(B)} , \quad B \subset \Omega \text{ open and bounded}
\]

the existence of, for example, continuous trace operators and embedding theorems holds as for classical Sobolev spaces (see Appendix A).

The following theorem contains the aforementioned extension of Weyl’s Lemma. In order to later use the theorem in a bootstrapping-type argument, we express both the regularity requirements and the resulting properties using the space \( H^{m}_{2,B-loc}(\Omega) \).

**Theorem 3.4.** Let \( \Omega \subset \mathbb{R}^n \) be a domain with a \( C^{m+k} \)-boundary. Let \( F \in (H^m_2(\Omega))^* \) be a functional satisfying

\[
\langle F, \psi \rangle := \int_\Omega \sum_{|\alpha| \leq m} g_\alpha(x) D^\alpha \psi(x) \, dx \quad \forall \psi \in H^m_2(\Omega), \supp \psi \text{ bounded}
\]

with

\[
g_\alpha \in H^{\max(0,k+|\alpha|-m)}_{2,B-loc}(\Omega) \quad \forall |\alpha| \leq m .
\]

Let \( a(\cdot,\cdot) \) be a \( H^m_2(\Omega) \)-coercive bilinear form of type

\[
a(u,v) := \int_\Omega \sum_{|\alpha|,|\beta| \leq m} a_{\alpha\beta}(x) D^\alpha u \, D^\beta v \, dx \quad \forall u,v \in H^m_2(\Omega)
\]

with coefficients

\[
a_{\alpha\beta} \in C^k(\overline{\Omega}) \quad \forall |\alpha|,|\beta| \leq m .
\]

If \( u \in H^m_2(\Omega) \) satisfies

\[
a(u,\psi) = \langle F, \psi \rangle \quad \forall \psi \in H^m_2(\Omega)
\]

then \( u \in H^{m+k}_{2,B-loc}(\Omega) \).
Proof.

Let \( \chi \in C^\infty_c(\mathbb{R}^n) \). For each \( x \in \Omega \) choose an open bounded neighborhood \( U_x \subset \subset \Omega \) of \( x \). In addition, for each \( x \in \partial \Omega \) choose an open bounded neighborhood \( U_x \) of \( x \) and a \((k + m)\)-diffeomorphism \( \Phi_x : U_x \to W \) with \( \Phi_x(U_x \cap \Omega) = W^+ \) and \( \Phi_x(U_x \cap \partial \Omega) = \{ x \in W \mid x_n = 0 \} \). By compactness of \( \text{supp} \chi \), one can find finitely many \( U_{x_i}, i = 1, \ldots, l \) such that \( \text{supp} \chi \subset \bigcup_{i=1}^l U_{x_i} \). Let \( \{ \chi_i \}_{i=1}^l \) be a partition of unity subordinate to \( \{ U_{x_i} \}_{i=1}^l \) with \( \sum_{i=1}^l \chi_i = 1 \) on \( \text{supp} \chi \).

Consider \( x_i, i = 1, \ldots, l \) with \( x_i \in \Omega \). By (3.68), \( u \) satisfies

\[
(3.69) \quad a(u, \psi) = \int_{U_{x_i}} \sum_{|\alpha|, |\beta| \leq m} a_{\alpha\beta}(x) \, D^\alpha u \, D^\beta \psi \, dx = \int_{U_{x_i}} \sum_{|\alpha| \leq m} g_\alpha(x) \, D^\alpha \psi(x) \, dx
\]

for all \( \psi \in \hat{H}_2^m(U_{x_i}) \). By partial integration and assumption (3.65), one can write the integral above as

\[
(3.70) \quad a(u, \psi) = \int_{U_{x_i}} \sum_{|\alpha| \leq m-k} \bar{g}_\alpha(x) \, D^\alpha \psi(x) \, dx \quad \forall \psi \in \hat{H}_2^m(U_{x_i})
\]

with \( \bar{g}_\alpha \in L^2(U_{x_i}) \) for all \( \alpha \). Since \( a(\cdot, \cdot) \) is \( \hat{H}_2^m(U_{x_i}) \)-coercive, standard regularity theory implies \( \chi_i \chi u \in \hat{H}_2^{m+k}(U_{x_i}) \). Extending by 0 on \( \Omega \setminus U_{x_i} \) implies \( \chi_i \chi u \in H_2^{m+k}(\Omega) \).

Now consider \( x_i, i = 1, \ldots, l \) with \( x_i \in \partial \Omega \). Put \( \Phi = \Phi_{x_i} \) and let \( \ast \Phi : H_2^m(W^+) \to H_2^m(U_{x_i} \cap \Omega) \) denote the associated pull-back operator. Define on \( V \times V \) the bilinear form

\[
(3.71) \quad A(\varphi, \psi) := a(\ast \Phi \varphi, \ast \Phi \psi) \quad \forall \varphi, \psi \in V
\]

and on \( V \) the functional

\[
(3.72) \quad \langle \mathcal{F}, \psi \rangle := \langle F, \ast \Phi \psi \rangle \quad \forall \psi \in V.
\]

By definition of \( V \), \( \ast \Phi \varphi = 0 \) in the trace sense on \( \partial U_{x_i} \cap \Omega \) for all \( \varphi \in V \). Hence putting \( \ast \Phi \varphi = 0 \) on \( \Omega \setminus U_{x_i} \), \( \ast \Phi \varphi \) extends to a function in \( H_2^m(\Omega) \) with bounded support. It follows that \( A \) and \( \mathcal{F} \) are well defined. It will now be verified that \( A, \mathcal{F}, \) and \( u \) satisfy the conditions of Lemma 3.3. The continuity of \( \ast \Phi^{-1} \) and \( \ast \Phi \) ensures the \( V \)-coerciveness of \( A \).

Furthermore, it can easily be verified that \( A \) is of type (3.6) with coefficients in \( C^k(\overline{W^+}) \). The continuity of \( \ast \Phi \) also implies \( \mathcal{F} \in V^* \). Now let \( \psi \in V \). One has

\[
(3.73) \quad \langle \mathcal{F}, \psi \rangle = \int_\Omega \sum_{|\alpha| \leq m} g_\alpha(x) \, D^\alpha [\ast \Phi \psi] \, dx
\]

\[
= \int_{U_{x_i} \cap \Omega} \sum_{|\alpha| \leq m} g_\alpha(x) \sum_{|\lambda| \leq |\alpha|} b_\lambda(x) \, D^\lambda \psi(\Phi(x)) \, dx
\]

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with functions $b_\lambda \in C^k(U_{x_i} \cap \Omega)$. It follows that

$$\langle F, \psi \rangle = \int_{U_{x_i} \cap \Omega} \sum_{|\alpha| \leq m} \tilde{g}_\alpha(x) D^\alpha \psi(\Phi(x)) \, dx$$

with $\tilde{g}_\alpha \in H_2^{\max(0,k+|\alpha|-m)}(U_{x_i} \cap \Omega)$. A change of variables thus yields

$$\langle F, \psi \rangle = \int_{W^+} \sum_{|\alpha| \leq m} \tilde{g}_\alpha(\Phi^{-1}(x)) |\det D\Phi^{-1}| D^\alpha \psi(x) \, dx$$

Consequently, one sees that $F$ is of type (3.8) and satisfies condition (3.38) of Lemma 3.3. Finally, it must now be shown that condition (3.40) is satisfied. Choose $\varepsilon > 0$ so small that $\text{supp} \chi_i \subset \Phi^{-1}(W_{3\varepsilon})$. Let $\eta \in C_c^\infty(U_{x_i})$ with $\eta = 1$ on $\Phi^{-1}(W_\varepsilon)$. It follows that $\Phi^{-1}(\eta u) \in V$ and $\chi_i \eta u = \chi_i u$. Furthermore, for $\psi \in \mathcal{V}$

$$\mathcal{A}(\Phi^{-1}(\eta u), \psi) = a(\eta u, \Phi \psi) = a(u, \Phi \psi) = \langle F, \Phi \psi \rangle = \langle F, \psi \rangle.$$

Hence $\mathcal{A}$, $F$, and $\Phi^{-1}(\eta u)$ satisfy condition (3.40). By Lemma 3.3, it now follows that $\hat{\chi} \Phi^{-1}(\eta u) \in H_2^{m+k}(W^+)$ for all $\hat{\chi} \in C_c^\infty(W_{3\varepsilon})$. Consequently $\hat{\chi} \eta u \in H_2^{m+k}(U_{x_i} \cap \Omega)$ for all $\hat{\chi} \in C_c^\infty(\Phi^{-1}(W_{3\varepsilon}))$. Choosing $\hat{\chi} = \chi \chi$ implies $\chi_i \chi u \in H_2^{m+k}(U_{x_i} \cap \Omega)$. Since $\chi_i \chi u = 0$ on $\partial U_{x_i} \cap \Omega$, extension by 0 yields $\chi_i \chi u \in H_2^{m+k}(\Omega)$. We now have

$$\chi u = \sum_{i=1}^l \chi_i \chi u \in H_2^{m+k}(\Omega) \ .$$

Remark 3.5. If $\Omega$ is bounded, $H_2^{m+k,2}_{B-loc}(\Omega) = H_2^{m+k}(\Omega)$. Thus Theorem 3.4 in this case implies global regularity.

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As a special case of Theorem 3.4 we now establish regularity for solutions of the generalized weak non-homogeneous Neumann problem (3.5).

Theorem 3.6. Let $\Omega \subset \mathbb{R}^n$ be a domain with a $C^{m+k+1}$-boundary. Let $B \in (H_2^m(\Omega))^*$ be a functional of the type

$$\langle B, \psi \rangle := \int_{\partial \Omega} \sum_{|\alpha| \leq m-1} b_\alpha(y) D^\alpha \psi \, dS(y) \ \forall \psi \in H_2^m(\Omega), \supp \psi \text{ bounded}$$

with

$$b_\alpha \in H_2^{\max\left(1, k+|\alpha|-m, \frac{3}{2}\right)}(\partial \Omega) \ \forall |\alpha| \leq m-1 \ .$$
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Let \( a(\cdot, \cdot) \) be a \( H^m_2(\Omega) \)-coercive bilinear form of type

\[
a(\varphi, \psi) := \int_\Omega \sum_{|\alpha|,|\beta| \leq m} a_{\alpha\beta}(x) D^\beta \varphi \, D^\alpha \psi \, dx \quad \forall \varphi, \psi \in H_2^m(\Omega)
\]

with

\[
a_{\alpha\beta} \in C^k(\overline{\Omega}) \quad \forall |\alpha|, |\beta| \leq m.
\]

If \( u \in H_2^m(\Omega) \) satisfies

\[
a(u, \psi) = \langle B, \psi \rangle \quad \forall \psi \in H_2^m(\Omega)
\]

then \( u \in H_2^{m+k}_{2,B-loc}(\Omega) \).

**Proof.**

Fix \( \alpha \) with \( |\alpha| \leq m - 1 \) and \( k + |\alpha| - m \geq 0 \). Consider the trace-operator

\[
T_{\alpha,k} : H_{2,B-loc}^{k+|\alpha|+m+2}(\Omega) \rightarrow H_{2,loc}^{k+|\alpha|+m+1}(\partial \Omega),
\]

\[
T_{\alpha,k} \varphi = \frac{\partial \varphi}{\partial n}\big|_{\partial \Omega} \quad \text{for} \ \varphi \in C^{k+|\alpha|+m+2}(\overline{\Omega})
\]

from Appendix A. The existence of a right inverse of \( T_{\alpha,k} \) (see Theorem A.8) implies that \( T_{\alpha,k} \) is onto. Hence there exists a \( g_\alpha \in H_{2,B-loc}^{k+|\alpha|+m+2}(\Omega) \) such that

\[
T_{\alpha,k} g_\alpha = b_\alpha.
\]

By Green’s Formula (see Theorem A.11), one has for any \( \psi \in H_2^m(\Omega) \) with bounded support the identity

\[
\int_{\partial \Omega} b_\alpha \, D^\alpha \psi \, dS = \int_{\partial \Omega} \frac{\partial g_\alpha}{\partial n} \, D^\alpha \psi \, dS = \int_{\Omega} Dg_\alpha \cdot D[D^\alpha \psi] \, dx + \int_{\Omega} \Delta g_\alpha \, D^\alpha \psi \, dx.
\]

Since the components of \( Dg_\alpha \) all belong to \( H_{2,B-loc}^{k+|\alpha|+1}(\Omega) \) and \( \Delta g_\alpha \in H_{2,B-loc}^{k+|\alpha|-m}(\Omega) \), it follows that

\[
\int_{\partial \Omega} b_\alpha(y) \, D^\alpha \psi \, dS(y) = \int_{\Omega} \sum_{|\tilde{\alpha}| \leq m} \tilde{g}_\tilde{\alpha}(x) \, D^{\tilde{\alpha}} \psi \, dx
\]

with \( \tilde{g}_\tilde{\alpha} \in H_{2,B-loc}^{k+|\alpha|-m}(\Omega) \).
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In the case $k + |\alpha| - m < 0$, $b_\alpha \in H^1_{2,\text{loc}}(\partial \Omega)$ by assumption. Using the right inverse of the trace operator

$$T_k : H^2_{2,B-\text{loc}}(\Omega) \to H^1_{2,\text{loc}}(\partial \Omega)$$

to find $g_\alpha \in H^2_{2,B-\text{loc}}(\Omega)$ with $\frac{\partial g_\alpha}{\partial n} = b_\alpha$, it follows as above that

$$\int_{\partial \Omega} b_\alpha \, D^\alpha \psi \, dS = \int_{\Omega} Dg_\alpha \cdot D[D^\alpha \psi] \, dx + \int_{\Omega} \Delta g_\alpha \, D^\alpha \psi \, dx$$

for $\psi \in H^2_2(\Omega)$ with $\text{supp} \psi$ bounded. Consequently

$$\int_{\partial \Omega} b_\alpha \, D^\alpha \psi \, dS = \int_{\Omega} \sum_{|\tilde{\alpha}| \leq m} \tilde{g}_{\tilde{\alpha}}(x) \, D^{\tilde{\alpha}} \psi \, dx$$

with $\tilde{g}_{\tilde{\alpha}} \in L^2_{B-\text{loc}}(\Omega) = H^{\max(0,k+|\tilde{\alpha}|-m)}_{2,B-\text{loc}}(\Omega)$.

Combining (3.83) and (3.84) now yields

$$\langle B, \psi \rangle = \int_{\Omega} \sum_{|\tilde{\alpha}| \leq m} \tilde{g}_{\tilde{\alpha}}(x) \, D^{\tilde{\alpha}} \psi \, dx \quad \forall \psi \in H^m_2(\Omega) \, , \, \text{supp} \psi \text{ bounded}$$

with

$$\tilde{g}_{\tilde{\alpha}} \in H^{\max(0,k+|\tilde{\alpha}|-m)}_{2,B-\text{loc}}(\Omega) \quad \forall |\tilde{\alpha}| \leq m \, .$$

Thus $B$ satisfies the conditions of Theorem 3.4 from which it follows that $u \in H^{m+k}_{2,B-\text{loc}}(\Omega)$.

Using the theorem above, regularity for the problem studied in Section 2 can now be proved. Since the problem in Section 2 was posed in the weighted Sobolev-space $H^1_2(\mathbb{R} \times \Omega, e^{-x})$ and not in a classical $H^m_2(\Omega)$-space, Theorem 3.6 cannot be applied directly. However, as will be seen in the following theorem, the weight function causes no significant difference with respect to the regularity properties.

**Theorem 3.7.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^{k+2}$-boundary $\Gamma$. Let $k \in \mathbb{N}$ and $b \in H^{k-\frac{1}{2}}_{2,\text{loc}}(\mathbb{R} \times \Gamma) \cap L^2(\mathbb{R} \times \Gamma, e^{-x})$. Assume $u \in H^1_2(\mathbb{R} \times \Omega, e^{-x})$ satisfies

$$\int_{\mathbb{R} \times \Omega} Du \cdot Dv \, e^{-x} \, d(x, y) = \int_{\mathbb{R} \times \Gamma} bv \, e^{-x} \, dS(y) \, dx \quad \forall v \in H^1_2(\mathbb{R} \times \Omega, e^{-x})$$

then $u \in H^{1+k}_{2,B-\text{loc}}(\mathbb{R} \times \Omega)$.

**Proof.**

Define on $H^1_2(\mathbb{R} \times \Omega, e^{-x})$ the bilinear form

$$a(v, w) := \int_{\mathbb{R} \times \Omega} Dw \cdot Dw \, e^{-x} \, d(x, y) \quad \forall v, w \in H^1_2(\mathbb{R} \times \Omega, e^{-x}) \, .$$
Clearly \( a(\cdot, \cdot) \) is bounded. Furthermore, using Lemma 2.1 one has
\[
a(v, v) = \int_{\Omega} |Dv|^2 e^{-x} \, dx, \quad (3.86)
\]
\[
\geq \frac{1}{5} \left( \int_{\Omega} v^2 e^{-x} \, dx + \int_{\Omega} |Dv|^2 e^{-x} \, dx \right) = \frac{1}{5} \|v\|_{H^1_2(\mathbb{R} \times \Omega, e^{-x})}^2.
\]
It follows that \( a(\cdot, \cdot) \) is \( H^1_2(\mathbb{R} \times \Omega, e^{-x}) \)-coercive. Now define on \( H^1_2(\mathbb{R} \times \Omega) \) the bilinear form
\[
A(\varphi, \psi) := a(\varphi e^{\frac{1}{2}x}, \psi e^{\frac{1}{2}x}) \quad \forall \varphi, \psi \in H^1_2(\mathbb{R} \times \Omega).
\]
By (3.86) one has
\[
A(\varphi, \varphi) = a(\varphi e^{\frac{1}{2}x}, \varphi e^{\frac{1}{2}x}) \geq \frac{1}{5} \left\| \varphi e^{\frac{1}{2}x} \right\|_{H^1_2(\mathbb{R} \times \Omega, e^{-x})}^2 \geq C \left\| \varphi \right\|_{H^1_2(\mathbb{R} \times \Omega)}^2
\]
for \( \varphi \in H^1_2(\mathbb{R} \times \Omega) \). Thus \( A \) is \( H^1_2(\mathbb{R} \times \Omega) \)-coercive. A simple calculation shows that \( A \) is of the type (3.80). Finally, define on \( H^1_2(\mathbb{R} \times \Omega) \) the functional
\[
\langle B, \psi \rangle := \int_{\mathbb{R} \times \Gamma} (b e^{-\frac{1}{2}x}) \psi \, dS(y) \, dx \quad \forall \psi \in H^1_2(\mathbb{R} \times \Omega).
\]
From (3.85) and the definition of \( A \) it follows that
\[
A(u e^{-\frac{1}{2}x}, \psi) = \langle B, \psi \rangle \quad \forall \psi \in H^1_2(\mathbb{R} \times \Omega).
\]
By assumption, \( b \in H^{k-\frac{1}{2}}_{2, loc}(\mathbb{R} \times \Gamma) \) and hence also \( b e^{-\frac{1}{2}x} \in H^{k-\frac{1}{2}}_{2, loc}(\mathbb{R} \times \Gamma) \). Thus all conditions of Theorem 3.6 are satisfied for \( u e^{-\frac{1}{2}x}, A, \) and \( B \). Hence Theorem 3.6 implies \( u e^{-\frac{1}{2}x} \in H^{1+k}_{2, B \text{-} loc}(\mathbb{R} \times \Omega) \) and thereby also \( u \in H^{1+k}_{2, B \text{-} loc}(\mathbb{R} \times \Omega) \).

Consider now a solution \( u \) of problem (1.3), which we solved in Section 2. Assuming \( f \in C^1(\mathbb{R}) \) with \( f' \) bounded we have \( f(u) \in H^1_2(\mathbb{R} \times \Omega, e^{-x}) \) and hence in the trace sense \( f(u) \in H^\frac{1}{2}_{2, loc}(\mathbb{R} \times \Gamma) \cap L^2(\mathbb{R} \times \Gamma, e^{-x}) \). By the theorem above, \( u \in H^2_{2, B \text{-} loc}(\mathbb{R} \times \Omega) \) follows. To the extent that this additional regularity of \( u \) translates into the same additional regularity of \( f(u) \), boot-strapping the argument would imply that \( u \) is ”as regular” as \( f \). More specifically, we have the following result.

**Theorem 3.8.** Let \( n \leq 3 \) and \( \Omega \subset \mathbb{R}^n \) be a bounded domain with a \( C^{k+2} \)-boundary. Let \( f \in C^k(\mathbb{R}) \) with all derivatives bounded,
\[
\|f^{(i)}\|_{\infty} \leq M \quad i = 1, \ldots, k.
\]
If \( u \in H^1_2(\mathbb{R} \times \Omega, e^{-x}) \) satisfies
\[
\int_{\mathbb{R} \times \Omega} Du \cdot Dv e^{-x} \, dx = \int_{\mathbb{R} \times \Gamma} f(u) v e^{-x} \, dS(y) \, dx \quad \forall v \in H^1_2(\mathbb{R} \times \Omega, e^{-x})
\]
then \( u \in H^{k+1}_{2, B \text{-} loc}(\mathbb{R} \times \Omega) \).
Proof.

As mentioned above, the boundedness of $f'$ implies $f(u) \in H^1_{2,\text{loc}}(\mathbb{R} \times \Gamma)$ and hence $u \in H^{2}_{2,\text{B–loc}}(\mathbb{R} \times \Omega)$ by Theorem 3.7. Thus the theorem holds for $k = 1$. Consider now $k > 1$ and assume the theorem has been proved for $k - 1$. Then $u \in H^k_{2,\text{B–loc}}(\mathbb{R} \times \Omega)$ by assumption. Now $u \in H^{k+1}_{2,\text{B–loc}}(\mathbb{R} \times \Omega)$ must be shown.

Consider a $k$'th order derivative

\begin{equation}
D^\alpha[f(u)] = \sum f^{(i)}(u) D^{\beta_1} u \cdots D^{\beta_k} u , \quad |\alpha| = k
\end{equation}

of $f(u)$. By the chain rule, a term in the sum above has one of the forms

1. $f'(u) D^\beta u$ , $|\beta| = k$
2. $f''(u) D^{\beta_1} u D^{\beta_2} u$ , $|\beta_1| = k - 1$ and $|\beta_2| = 1$
3. $f^{(i)}(u) D^{\beta_1} u \cdots D^{\beta_j} u$ , $|\beta_h| \leq k - 2$ for $h = 1, \ldots, j$ and $j \geq 2$.

Clearly the terms of type 1 belong to $L^1_{B–loc}(\mathbb{R} \times \Omega)$. Consider a term $f''(u) D^{\beta_1} u D^{\beta_2} u$ of type 2. One has both $D^{\beta_1} u \in H^1_{2,\text{B–loc}}(\mathbb{R} \times \Omega)$ and $D^{\beta_2} u \in H^1_{2,\text{B–loc}}(\mathbb{R} \times \Omega)$. Since $n \leq 3$ one has dim$(\mathbb{R} \times \Omega) \leq 4$ and thus the embedding

\[ H^1_{2,\text{B–loc}}(\mathbb{R} \times \Omega) \hookrightarrow L^4_{B–loc}(\mathbb{R} \times \Omega) \]

holds (see Theorem A.12). It follows that the product $D^{\beta_1} u \cdot D^{\beta_2} u \in L^2_{B–loc}(\mathbb{R} \times \Omega)$ and by the boundedness of $f''$ thus also $f''(u) D^{\beta_1} u \cdot D^{\beta_2} u \in L^2_{B–loc}(\mathbb{R} \times \Omega)$.

Finally consider a term $f^{(i)}(u) D^{\beta_1} u \cdots D^{\beta_j} u$ in the sum (3.89) of type 3. Since the highest order derivative occurring is less than $k - 2$, one has $D^{\beta_h} u \in H^2_{2,\text{B–loc}}(\mathbb{R} \times \Omega)$ for $h = 1, \ldots, j$. Since dim$(\mathbb{R} \times \Omega) \leq 4$ the embedding

\begin{equation}
H^2_{2,\text{B–loc}}(\mathbb{R} \times \Omega) \hookrightarrow L^q_{B–loc}(\mathbb{R} \times \Omega)
\end{equation}

holds for all $q \geq 2$ (see Theorem A.12). Putting $q = 2j$ in (3.90) and applying the Hölder inequality thus implies $D^{\beta_1} u \cdots D^{\beta_j} u \in L^2_{B–loc}(\mathbb{R} \times \Omega)$. By the boundedness of $f^{(i)}$ hence also $f^{(i)}(u) D^{\beta_1} u \cdots D^{\beta_j} u \in L^2_{B–loc}(\mathbb{R} \times \Omega)$.

We have now proved that every element in the sum (3.89) belongs to $L^2_{B–loc}(\mathbb{R} \times \Omega)$. Consequently $D^\alpha[f(u)] \in L^2_{B–loc}(\mathbb{R} \times \Omega)$. It follows that $f(u) \in H^k_{2,\text{B–loc}}(\mathbb{R} \times \Omega)$ and thereby $f(u) \in H^{k+\frac{1}{2}}_{2,\text{loc}}(\mathbb{R} \times \Gamma)$ in the trace sense. By Theorem 3.7 we can finally deduce that $u \in H^{k+1}_{2,\text{B–loc}}(\mathbb{R} \times \Omega)$. 

\section{Appendix}

\textbf{Theorem A.1.} Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^m$-boundary. Then $\mathbb{R} \times \Omega$ satisfies the uniform $C^m$-regularity condition.

\textbf{Proof.} \\
Since $\Omega$ is bounded with a $C^m$-boundary, we can find a finite open cover $\{U_j\}_{j=1}^k$ of $\partial \Omega$ and for each $U_j$ an associated $m$-smooth diffeomorphism

$$\Phi_j : U_j \to W^n := \{\xi \in \mathbb{R}^n \mid |\xi| < 1\}$$

satisfying the uniform $C^m$-regularity condition:

1. For some finite $R$, every collection of $R + 1$ of the sets $U_j$ has empty intersection.
2. For some $\delta > 0$, $\Omega_\delta \subset \bigcup_{j=1}^\infty \Phi_j^{-1}(\{\xi \in \mathbb{R}^n \mid |\xi| < \frac{1}{2}\})$ whereby $\Omega_\delta := \{y \in \Omega \mid \text{dist}(y, \partial \Omega) < \delta\}$.
3. For each $j$, $\Phi_j(U_j \cap \Omega) = \{\xi \in \mathbb{R}^n \mid \xi_n > 0\}$.
4. If $(\varphi_{j,1}, \ldots, \varphi_{j,n})$ and $(\varphi_{j,1}^{-1}, \ldots, \varphi_{j,n}^{-1})$ are the components of $\Phi_j$ and $\Phi_j^{-1}$, then there is a finite constant $M$ such that for every $\alpha$ with $0 \leq |\alpha| \leq m$, every $1 \leq i \leq n$, and every $j$ we have

$$|D^\alpha \varphi_{j,i}(y)| \leq M \quad \forall y \in U_j \quad \text{and}$$
$$|D^\alpha \varphi_{j,n}^{-1}(\xi)| \leq M \quad \forall \xi \in W^n.$$ 

Put

$$\Phi_{(j,i)}(x, y) := (x - (i + 1), \Phi_j(y)) \quad \forall (x, y) \in ((i, i + 2) \times U_j).$$

Clearly $\{(i, i + 2) \times U_j\}_{(j,i) \in \{1, \ldots, k\} \times \mathbb{Z}}$ is a locally finite open cover of $\mathbb{R} \times \partial \Omega$ and

$$\Phi_{(j,i)} : (i, i + 2) \times U_j \to W^{n+1}$$

a corresponding sequence of $m$-smooth diffeomorphisms. Furthermore, since $\{\Phi_j\}_{j=1}^k$ satisfies condition 1-4 with respect to $\Omega$, so does $\{\Phi_{(j,i)}\}_{(j,i) \in \{1, \ldots, k\} \times \mathbb{Z}}$ with respect to $\mathbb{R} \times \Omega$. 

\textbf{Theorem A.2.} Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the uniform cone condition. There exists a continuous extension operator

$$(A.1) \quad E : H^1_2(\mathbb{R} \times \Omega) \to H^1_2(\mathbb{R}^{n+1})$$

with the property

$$(A.2) \quad E(u) = u \quad a.e. \ in \ \mathbb{R} \times \Omega.$$
Proof. Let \( \{U_j\}_{j=1}^k \) be a locally finite open cover of \( \partial \Omega \) corresponding to the uniform cone condition. Put
\[
\tilde{U}_j := \mathbb{R} \times U_j .
\]
Clearly \( \{\tilde{U}_j\}_{j=1}^k \) is a locally finite open cover of \( \mathbb{R} \times \partial \Omega \). Since \( \{U_j\}_{j=1}^k \) satisfies the uniform cone condition, so does \( \{\tilde{U}_j\}_{j=1}^k \) with the only exception that \( \tilde{U}_j \) is not bounded. This, however, suffices in order to satisfy the conditions of the Calderón Extension Theorem (See Theorem 4.32 in [Ada75]) from which the existence of an extension operator follows.

**Definition A.3.** Let \( \Omega \subset \mathbb{R}^n \) be a domain and \( m \in \mathbb{N} \). We define \( H^m_{2,B-\text{loc}}(\Omega) \) as the space
\[
H^m_{2,B-\text{loc}}(\Omega) := \{ u \in L^2_{\text{loc}}(\Omega) \mid \chi u \in H^m_2(\Omega) \text{ for all } \chi \in C_\infty^\infty(\mathbb{R}^n) \}
\]
equipped with the topology induced by the family of seminorms
\[
\| \cdot \|_{B} := \| \| \cdot \|_{H^m_2(B)} , \quad B \subset \Omega \text{ open and bounded}.
\]

**Remark A.4.** It follows that \( H^m_{2,B-\text{loc}}(\Omega) \subset H^m_{2,\text{loc}}(\Omega) \) for general domains and \( H^m_{2,B-\text{loc}}(\Omega) = H^m_2(\Omega) \) for bounded domains.

**Theorem A.5.** Let \( m \geq 1 \) and \( \Omega \subset \mathbb{R}^n \) be a domain with \( C^m \)-boundary. There exists a continuous linear trace operator
\[
T_0 : H^m_{2,B-\text{loc}}(\Omega) \to H^{m-\frac{1}{2}}_{2,\text{loc}}(\partial \Omega)
\]
with the property
\[
T_0 \varphi = \varphi_{|\partial \Omega} , \quad \forall \varphi \in C^m(\overline{\Omega}) .
\]

**Proof.**
We will construct \( T_0 \) as in the proof of Theorem 8.7 in [Wlo87]. Since \( \Omega \) has a \( C^m \)-boundary, we can find a locally finite open cover of bounded subsets \( \{U_j\}_{j=1}^\infty \) of \( \partial \Omega \) and corresponding \( m \)-diffeomorphisms \( \{\Phi_j\}_{j=1}^\infty \) satisfying
\[
\begin{align*}
\Phi_j : & U_j \to W^n , \\
\Phi_j(U_j \cap \partial \Omega) &= \{ \xi \in W^n \mid \xi_n = 0 \} (= W^{n-1}) , \text{ and} \\
\Phi_j(U_j \cap \Omega) &= \{ \xi \in W^n \mid \xi_n > 0 \} (= W^n_+) .
\end{align*}
\]
Let \( \{\alpha_j\}_{j=1}^\infty \) be a partition of unity subordinate to \( \{U_j\}_{j=1}^\infty \). Furthermore, let \( \mathcal{F} \) denote the extension operator
\[
\mathcal{F} : H^m_2(\mathbb{R}^n_) \to H^m_2(\mathbb{R}^n)
\]
and \( T_0 \) the trace operator
\[
T_0 : \dot{H}^m_2(W^n) \to \dot{H}^{m-\frac{1}{2}}(W^{n-1}).
\]

We have the diagram
\[
\begin{array}{cccc}
H^m_{2,B-loc}(\Omega) & \xrightarrow{\alpha_j} & H^m_2(U_j \cap \Omega) & \xrightarrow{\ast \Phi_j^{-1}} & H^m_2(W_+) & \xrightarrow{\mathcal{F}} & \dot{H}^m_2(W^n) & \xrightarrow{T_0} \\
\dot{H}^{m-\frac{1}{2}}(W^{n-1}) & \xrightarrow{\ast \Phi_j} & \dot{H}^{m-\frac{1}{2}}(U_j \cap \partial \Omega) & \to & H^{m-\frac{1}{2}}(\partial \Omega).
\end{array}
\]

Define now
\[
(A.7) \quad T_0 \varphi := \sum_{j=1}^{\infty} \ast \Phi_j \circ T_0 \circ \mathcal{F} \circ \ast \Phi_j^{-1}(\alpha_j \varphi) \quad \forall \varphi \in H^m_{2,B-loc}(\Omega).
\]

Since the cover \( \{U_j\}_{j=1}^{\infty} \) is locally finite, \( T_0 \varphi \) is well-defined pointwise a.e.

Consider an open subset \( K \subset \subset \partial \Omega \) of \( \partial \Omega \). Since \( K \) is compact, only a finite number of the \( \alpha_j \)'s, say \( \alpha_1, \ldots, \alpha_k \), do not vanish on \( K \). Put \( B = \cup_{j=1}^{k} U_j \cap \Omega \). By continuity of the operators occurring in sum on the right-hand side of (A.7) it follows that
\[
(A.8) \quad \|T_0 \varphi\|_{H^m_{2,B-loc}(\partial \Omega)} = \left\| \sum_{j=1}^{k} \ast \Phi_j \circ T_0 \circ \mathcal{F} \circ \ast \Phi_j^{-1}(\alpha_j \varphi) \right\|_{H^m_{2,B-loc}(\partial \Omega)} \leq C_K \|\varphi\|_{H^m_2(B)}.
\]

Since \( B \) is bounded, we see from (A.8) that for any sequence \( \{u_n\}_{n=1}^{\infty} \subset H^m_{2,B-loc}(\Omega) \) with \( u_n \to u \) in \( H^m_{2,B-loc}(\Omega) \) holds \( T_0 u_n \to T_0 u \) in \( H^{m-\frac{1}{2}}(K) \). Thus \( T_0 \) is continuous from \( H^m_{2,B-loc}(\Omega) \) into \( H^{m-\frac{1}{2}}(\partial \Omega) \).

Property (A.6) follows as in the proof of Theorem 8.7 in [Wlo87].

**Theorem A.6.** Let \( m \geq 1 \) and \( \Omega \subset \mathbb{R}^n \) be a domain with \( C^m \)-boundary. There exists a continuous linear operator
\[
(A.9) \quad Z_0 : \dot{H}^{m-\frac{1}{2}}_{2,loc}(\partial \Omega) \to H^m_{2,B-loc}(\Omega)
\]
with the property
\[
(A.10) \quad T_0 \circ Z_0 = \text{id}.
\]

**Proof.**

Constructing \( Z_0 \) as in the proof of Theorem 8.8 in [Wlo87] it follows by a similar argument as in the proof of Theorem A.5 that \( Z_0 \) is continuous from \( H^{m-\frac{1}{2}}_{2,loc}(\partial \Omega) \) into \( H^m_{2,B-loc}(\Omega) \). Property (A.10) follows as in Theorem 8.8 in [Wlo87].

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**Theorem A.7.** Let \( m \geq 2 \) and \( \Omega \subset \mathbb{R}^n \) be a domain with \( C^{m+1} \)-boundary. There exists a continuous linear trace operator

\[ T_1 : H_{2,B-loc}^m(\Omega) \to H_{2,loc}^{m-\frac{4}{n}}(\partial \Omega) \]

with the property

\[ T_1 \varphi = \frac{\partial \varphi}{\partial n} \big|_{\partial \Omega} \quad \forall \varphi \in C^m(\overline{\Omega}). \]

**Proof.**

Follows from Theorem 8.7 in [Wlo87] in a similar way as Theorem A.5.

**Theorem A.8.** Let \( m \geq 2 \) and \( \Omega \subset \mathbb{R}^n \) be a domain with \( C^{m+1} \)-boundary. There exists a continuous linear operator

\[ Z_1 : H_{2,loc}^{m-\frac{3}{n}}(\partial \Omega) \to H_{2,B-loc}^m(\Omega) \]

with the property

\[ T_1 \circ Z_1 = \text{id}. \]

**Proof.**

Follows from Theorem 8.8 in [Wlo87] similar to Theorem A.6.

**Theorem A.9.** Let \( \Omega \subset \mathbb{R}^n \) be a domain with \( C^m \)-boundary. Then \( C^m(\overline{\Omega}) \) lies dense in \( H_{2,B-loc}^m(\Omega) \),

\[ H_{2,B-loc}^m(\Omega) = \overline{C^m(\overline{\Omega})}^{H_{2,B-loc}^m(\Omega)}. \]

**Proof.**

This follows by the same arguments as in the classical proof by Meyers and Serrin that \( C^m(\overline{\Omega}) \) lies dense in \( H_2^m(\Omega) \) (see for example Theorem 3.5 in [Wlo87]).

**Remark A.10.** Throughout the work we have breached the classical notational convention of using \( H_2^m(\Omega) \) to denote the closure of \( C^\infty(\overline{\Omega}) \) in the Sobolev norm. Instead, we have defined \( H_2^m(\Omega) \) as the subspace of \( L^2(\Omega) \) with weak derivatives again in \( L^2(\Omega) \), which in the classical notation is denoted \( W_2^m(\Omega) \). Since \( H_2^m(\Omega) = W_2^m(\Omega) \) and by the above also \( H_{2,B-loc}^m(\Omega) = W_{2,B-loc}^m(\Omega) \), this, however, should cause no confusion.
Theorem A.11. Let $\Omega$ be a domain with $C^2$-boundary. Let $u \in H^2_{2,B-loc}(\Omega)$. For $\psi \in H^1_2(\Omega)$ with $\text{supp} \, \psi$ bounded, the Green's Formula

\begin{equation}
\int_{\Omega} Du \cdot D\psi \, dx = \int_{\partial \Omega} \frac{\partial u}{\partial n} \psi \, dS(y) - \int_{\Omega} \Delta u \, \psi \, dx \tag{A.16}
\end{equation}

holds.

Proof.

Since $\text{supp} \, \psi$ is bounded the integrals in (A.16) reduce to integrals over a bounded set $B$. Since $u \in H^2_2(B)$, the Green’s Formula follows. \hfill $\Box$

Theorem A.12. Let $\Omega$ be a domain in $\mathbb{R}^n$ satisfying the cone condition. The following embeddings hold

\begin{align}
&2m \geq n : \quad H^m_{2,B-loc}(\Omega) \hookrightarrow L^q_{B-loc}(\Omega) \quad \text{for } 2 \leq q \tag{A.17} \\
&2m < n : \quad H^m_{2,B-loc}(\Omega) \hookrightarrow L^q_{B-loc}(\Omega) \quad \text{for } 2 \leq q \leq \frac{2n}{n-2m} \tag{A.18}
\end{align}

Here $L^q_{B-loc}(\Omega)$ is defined as the space

\begin{equation}
L^q_{B-loc}(\Omega) := \{ u \in L^q_{loc}(\Omega) \mid \chi u \in L^q(\Omega) \text{ for all } \chi \in C^\infty_c(\mathbb{R}^n) \} \tag{A.19}
\end{equation}

equipped with the topology induced by the family of seminorms

\begin{equation}
\| \cdot \|_B := \| \cdot \|_{L^q(B)} , \quad B \subset \Omega \text{ open and bounded} \tag{A.20}
\end{equation}

Proof.

Follows by localization and the classical Sobolev Embedding Theorem. \hfill $\Box$
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